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## ► To cite this version:

Erik Burman, Alexandre Ern. An unfitted Hybrid High-Order method for elliptic interface problems. SIAM Journal on Numerical Analysis, 2018, 56 (3), pp.1525-1546. hal-01625421v2

**HAL Id: hal-01625421**

**<https://hal.science/hal-01625421v2>**

Submitted on 19 Mar 2018

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# An unfitted Hybrid High-Order method for elliptic interface problems\*

Erik Burman<sup>†</sup>      Alexandre Ern<sup>‡</sup>

March 18, 2018

## Abstract

We design and analyze a Hybrid High-Order (HHO) method on unfitted meshes to approximate elliptic interface problems. The curved interface can cut through the mesh cells in a very general fashion. As in classical HHO methods, the present unfitted method introduces cell and face unknowns in uncut cells, but doubles the unknowns in the cut cells and on the cut faces. The main difference with classical HHO methods is that a Nitsche-type formulation is used to devise the local reconstruction operator. As in classical HHO methods, cell unknowns can be eliminated locally leading to a global problem coupling only the face unknowns by means of a compact stencil. We prove stability estimates and optimal error estimates in the  $H^1$ -norm. Robustness with respect to cuts is achieved by a local cell-agglomeration procedure taking full advantage of the fact that HHO methods support polyhedral meshes. Robustness with respect to the contrast in the material properties from both sides of the interface is achieved by using material-dependent weights in Nitsche's formulation.

## 1 Introduction

The Hybrid High-Order (HHO) method has been recently introduced in [15] for linear elasticity problems and in [16] for diffusion problems. The HHO method is formulated in terms of cell and face unknowns. The cell unknowns can be eliminated locally by using a Schur complement technique (also known as static condensation), leading to a global transmission problem coupling only the face unknowns by means of a compact stencil. The HHO method is devised locally from two ingredients: a reconstruction operator and a stabilization operator. This leads to a discretization method that supports general meshes (with possible polyhedral cells and non-matching interfaces), is locally conservative and delivers energy-norm error estimates of order  $(k+1)$  (and  $L^2$ -norm error estimates of order  $(k+2)$  under full elliptic regularity) if polynomials of order  $k \geq 0$  are used for the face unknowns. As shown in [13], the HHO method can be fitted into the family of Hybridizable Discontinuous Galerkin (HDG) methods introduced in [12] and is closely related to the nonconforming Virtual Element Method (ncVEM) studied in [1].

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\*Draft version, March 18, 2018

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The use of polyhedral meshes can greatly simplify the meshing of complicated geometries. Nevertheless, in some situations, it is still convenient to avoid the meshing of boundaries and internal interfaces. This is the case when the boundary changes during the computation, such as in free-boundary and optimization problems, and when the boundary or the internal interface is curved. In this paper, we are interested in devising a high-order approximation method for elliptic interface problems. To handle difficulties with curved interfaces in classical finite element methods, boundary-penalty methods [2, 3] have been proposed, where the computational mesh does not need to respect the interface. In order to improve the accuracy, unfitted finite element methods were introduced in [23] drawing on the seminal ideas of Nitsche [29] for the weak imposition of boundary conditions. The key idea is to design the finite element space so that singularities over the interface can be represented by a pair of polynomials in the cut cells. Similar approaches were then proposed in the context of discontinuous Galerkin methods in [4, 28, 25].

A well-known difficulty for unfitted finite element methods is that the conditioning of the resulting linear system has a strong dependence on how the interface cuts the mesh cells. This means that for unfavorable cuts, Nitsche’s formulation can be severely ill-conditioned. This difficulty has been solved in [23] by using weighted coupling terms with cut-dependent weights. However, there is a lack of robustness when the material properties (e.g., the diffusivities on each side of the interface) are highly contrasted. Robustness with respect to the contrast can be achieved by using material-dependent weights, as proposed in different contexts in [7, 19, 9], and in this case, a different mechanism is needed to handle unfavorable cuts. In the case of  $H^1$ -conforming methods, this problem can be overcome by adding a penalty term that weakly couples the polynomial approximation in adjacent cells as proposed in [5]. When using a discontinuous Galerkin approximation, another approach was proposed in [25] for fictitious domain problems where mesh cells with unfavorable cuts are merged with neighboring elements having a favorable cut. This idea is also explored in [24] for interface problems approximated by conforming finite elements on quadrilateral meshes whereby cells with an unfavorable cut are merged with adjacent quadrilateral cells (thus creating hanging nodes).

The so-called cutFEM framework was developed recently in [8] so as to couple different physical models over unfitted interfaces and to discretize PDEs over unfitted embedded submanifolds. The high-order approximation of the geometry of the interface was considered recently in [10] using a boundary correction based on local Taylor expansions and in [26] using an iso-parametric technique, the common objective being to simplify the numerical integration on domains with curved boundaries by allowing a piecewise affine representation of the interface. The cutFEM paradigm has also been applied to a variety of complex flow problems, see, e.g., [27], the recent PhD thesis [30], and references therein. A conforming finite element method with local remeshing in subcells, effectively fitting the mesh to the interface, followed by elimination of the local degrees of freedom, was introduced in [20].

The goal of the present work is to devise and analyze an HHO method using unfitted meshes. The approach consists of doubling the unknowns in the cut cells and the cut faces, in a spirit similar to unfitted finite element methods. For brevity, we only consider elliptic interface problems, but the material can be readily adapted to treat the (simpler) case of fictitious domain problems; such an adaptation is briefly reported in [6]. Our approach

combines the ideas of HHO methods (and more broadly HDG methods) with those from [23] concerning Nitsche's formulation, but with material-dependent weights rather than cut-dependent weights, and those from [25] to handle unfavorable cuts by a local cell-agglomeration procedure. The cell-agglomeration procedure takes full advantage of the fact that the HHO method supports general meshes with polyhedral cells. The resulting unfitted HHO method is robust with respect to the cuts and to the material properties. Our stability and error analysis of the unfitted HHO method sheds some novel light in the analysis of HHO methods. On the one hand, the local reconstruction operator is based on Nitsche's formulation and cannot be related, as in classical HHO methods, to a local elliptic projector. On the other hand, the error is measured by using some projector that is somewhat more elaborate than the local  $L^2$ -orthogonal projector used in classical HHO methods. Our main result is an  $H^1$ -error estimate of order  $(k+1)$  if polynomials of order  $k \geq 0$  are used for the face unknowns and polynomials of order  $(k+1)$  are used for the cell unknowns. We observe that we do not consider here cell unknowns of order  $k$  as in classical HHO methods. The overhead induced by this modification is marginal since, as usual, all the cell unknowns can be eliminated locally. Finally, we mention the recent numerical work combining the HDG method with the X-FEM technique for fictitious domain [21] and elliptic interface [22] problems. The main differences with the present unfitted HHO method is that we do not introduce unknowns at the interface (but rather double the unknowns at the mesh faces cut by the interface) and that we provide a thorough analysis including robustness with respect to cuts and contrast.

This paper is organized as follows. In Section 2, we introduce the elliptic interface problem we want to approximate. In Section 3, we present the discrete setting, including our main notation for the cut cells and the two assumptions we require on the mesh, and we prove two key trace inequalities under these assumptions. In Section 4, we present the unfitted HHO method. In Section 5, we present our stability and error analysis; our main result is Theorem 13. Finally, in Section 6 we show how the two mesh properties introduced in Section 3 can be satisfied by using a local cell-agglomeration procedure (under the assumption that the mesh is fine enough to resolve the interface). Computational results will be reported in a separate publication.

## 2 Model problem

Let  $\Omega$  be a domain in  $\mathbb{R}^d$  (open, bounded, connected, Lipschitz subset) and consider a partition of  $\Omega$  into two disjoint subdomains so that  $\overline{\Omega} = \overline{\Omega^1} \cup \overline{\Omega^2}$  with the interface  $\Gamma = \partial\Omega^1 \cap \partial\Omega^2$ . The unit normal vector  $\mathbf{n}_\Gamma$  to  $\Gamma$  conventionally points from  $\Omega^1$  to  $\Omega^2$ . For a smooth enough function defined on  $\Omega$ , we define its jump across  $\Gamma$  as  $\llbracket v \rrbracket_\Gamma := v|_{\Omega^1} - v|_{\Omega^2}$ . We consider the following interface problem:

$$-\nabla \cdot (\kappa \nabla u) = f, \quad \text{in } \Omega^1 \cup \Omega^2, \quad (1a)$$

$$\llbracket u \rrbracket_\Gamma = g_D, \quad \text{on } \Gamma, \quad (1b)$$

$$\llbracket \kappa \nabla u \rrbracket_\Gamma \cdot \mathbf{n}_\Gamma = g_N, \quad \text{on } \Gamma, \quad (1c)$$

$$u = 0 \quad \text{on } \partial\Omega, \quad (1d)$$

with  $f \in L^2(\Omega)$ ,  $g_D \in H^{\frac{1}{2}}(\Gamma)$ ,  $g_N \in L^2(\Gamma)$ . For simplicity we consider a homogeneous Dirichlet condition on  $\partial\Omega$ . To avoid technicalities, we assume that the diffusion coefficient  $\kappa$  is scalar-valued and that  $\kappa^i := \kappa|_{\Omega^i}$  is constant for each  $i \in \{1, 2\}$ . Without loss of generality, we assume that the numbering of the two subdomains is such that  $\kappa^1 < \kappa^2$ . In the rest of the paper, we assume that the interface  $\Gamma$  is a smooth  $(d-1)$ -dimensional manifold of class  $C^2$  that is not self-intersecting. This assumption can be relaxed at the price of additional technical issues that are not explored herein.

### 3 Discrete setting

We assume that the domain  $\Omega$  is a polyhedron with planar faces in  $\mathbb{R}^d$ . Let  $(\mathcal{T}_h)_{h>0}$  be a shape-regular family of matching meshes covering  $\Omega$  exactly. The meshes can have cells that are polyhedra with planar faces in  $\mathbb{R}^d$ , and hanging nodes are also possible. The mesh cells are considered to be open subsets of  $\mathbb{R}^d$ . For a subset  $S \subset \mathbb{R}^d$ ,  $h_S$  denotes the diameter of  $S$ , and for a mesh  $\mathcal{T}_h$ , the index  $h$  refers to the maximal diameter of the mesh cells. The shape-regularity criterion for polyhedral meshes is that they admit a matching simplicial sub-mesh that satisfies the usual shape-regularity criterion in the sense of Ciarlet and such that each sub-cell (resp., sub-face) belongs to only one mesh cell (resp., at most one mesh face). The shape-regularity of the mesh sequence is quantified by a parameter  $\rho \in (0, 1)$  (see Section 6 for further insight). In what follows,  $B(\mathbf{y}, a)$  denotes the open ball with center  $\mathbf{y}$  and radius  $a$ ,  $d(\mathbf{y}, A)$  denotes the distance of the point  $\mathbf{y}$  to the set  $A$ , and  $d(A, A')$  denotes the Hausdorff distance between the two sets  $A, A'$ .

#### 3.1 Main notation for unfitted meshes

Since the meshes are not fitted to the subsets  $\Omega^1$  and  $\Omega^2$ , there are mesh cells in  $\mathcal{T}_h$  that are cut by the interface  $\Gamma$ . Let us define the partition  $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^\Gamma \cup \mathcal{T}_h^2$ , where the subsets

$$\mathcal{T}_h^i := \{T \in \mathcal{T}_h \mid T \subset \Omega^i\}, \quad \forall i \in \{1, 2\}, \quad (2a)$$

$$\mathcal{T}_h^\Gamma := \{T \in \mathcal{T}_h \mid \text{meas}_{d-1}(T \cap \Gamma) > 0\}, \quad (2b)$$

collect, respectively, the mesh cells inside the subdomain  $\Omega^i$ ,  $i \in \{1, 2\}$ , and the mesh cells cut by the interface  $\Gamma$ . For any mesh cell  $T \in \mathcal{T}_h^\Gamma$  cut by the interface, we define

$$T^i := T \cap \Omega^i, \quad T^\Gamma := T \cap \Gamma. \quad (3)$$

The boundary of the sub-cell  $T^i$  is decomposed as follows:

$$\partial T^i = (\partial T)^i \cup T^\Gamma, \quad (4)$$

with the notation  $(\partial T)^i = \partial T \cap \Omega^i$ . For any mesh cell  $T \in \mathcal{T}_h$ , the set  $\mathcal{F}_{\partial T}$  collects the mesh faces located at the boundary  $\partial T$  of  $T$ . Whenever  $T \in \mathcal{T}_h^\Gamma$ , we consider the set

$$\mathcal{F}_{(\partial T)^i} = \{F^i = F \cap \Omega^i \mid F \in \mathcal{F}_{\partial T}, \text{meas}_{d-1}(F^i) > 0\}. \quad (5)$$

The sub-faces in  $\mathcal{F}_{(\partial T)^i}$  form a partition of  $(\partial T)^i$  (but not of  $\partial T^i$  since  $T^\Gamma$  is not included in  $\mathcal{F}_{(\partial T)^i}$ ). The notation is illustrated in Figure 1. Since the interface  $\Gamma$  is not self-intersecting

and smooth, there exists a length scale  $\ell_0$  so that, for all  $\mathbf{s} \in \Gamma$ , the subset  $\Gamma \cap B(\mathbf{s}, \ell_0)$  has only one connected component. In what follows, we assume that the mesh is fine enough so that  $h \leq \ell_0$ . This assumption implies that  $T^\Gamma$  has a single connected component, and that the sub-cells  $T^1$  and  $T^2$  are connected. We also assume that  $d(\Gamma, \partial\Omega) \geq 2h$ .

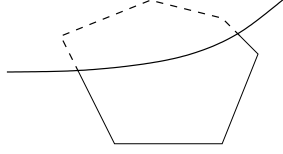


Figure 1: Hexagonal cell  $T$  cut by the interface  $\Gamma$ . The subdomain  $\Omega^1$  is located below  $\Gamma$ , and the subdomain  $\Omega^2$  is located above  $\Gamma$ .  $(\partial T)^1$  is shown using solid lines and  $(\partial T)^2$  using dashed lines; the sets  $\mathcal{F}_{(\partial T)^1}$  and  $\mathcal{F}_{(\partial T)^2}$  consist each of four elements, two of which are original faces of  $T$  and two of which are sub-faces of the two faces of  $T$  cut by the interface.

Let  $l \in \mathbb{N}$  be a polynomial degree and let  $S$  be an  $m$ -dimensional affine manifold in  $\Omega$  ( $m \leq d$ ); typically,  $S$  is a mesh (sub-)cell (so that  $m = d$ ) or a mesh (sub-)face (so that  $m = d - 1$ ). Then  $\mathbb{P}^l(S)$  denotes the space composed of the restriction to  $S$  of  $d$ -variate polynomials of degree at most  $l$ .

### 3.2 Mesh properties

We make the following two assumptions on the mesh. Assumption 1 means that the interface is properly described by the mesh; this assumption is quantified by an interface regularity parameter  $\gamma \in (0, 1)$ . Assumption 2 means that all the mesh cells are cut favorably by the interface; this property is quantified by a cut parameter  $\delta \in (0, 1)$ . We will show in Section 6 how to produce a shape-regular (polyhedral) mesh so that Assumption 1 and Assumption 2 hold true. The idea is that Assumption 1 can be satisfied by refining the mesh, whereas Assumption 2 can be satisfied by means of a local cell-agglomeration procedure.

**Assumption 1** (Resolving  $\Gamma$ ). *There is  $\gamma \in (0, 1)$  s.t. for all  $T \in \mathcal{T}_h^\Gamma$ , there is a point  $\hat{\mathbf{x}}_T \in \mathbb{R}^d$  so that, for all  $\mathbf{s} \in T^\Gamma$ ,  $\|\hat{\mathbf{x}}_T - \mathbf{s}\|_{\ell^2} \leq \gamma^{-1}h_T$  and  $d(\hat{\mathbf{x}}_T, T_\mathbf{s}\Gamma) \geq \gamma h_T$  where  $T_\mathbf{s}\Gamma$  is the tangent plane to  $\Gamma$  at the point  $\mathbf{s}$ .*

**Assumption 2** (Cut cells). *There is  $\delta \in (0, 1)$  such that, for all  $T \in \mathcal{T}_h^\Gamma$  and all  $i \in \{1, 2\}$ , there is  $\tilde{\mathbf{x}}_{T^i} \in T^i$  so that*

$$B(\tilde{\mathbf{x}}_{T^i}, \delta h_T) \subset T^i. \quad (6)$$

### 3.3 Trace inequalities

The purpose of Assumption 1 is to prove a multiplicative trace inequality that is needed to establish optimal approximation properties for the unfitted HHO method, whereas the purpose of Assumption 2 is to prove a discrete trace inequality that is needed in the stability analysis of the unfitted HHO method. Let us now prove these two trace inequalities.

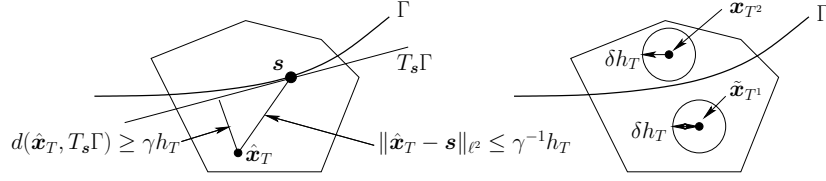


Figure 2: Illustration of Assumption 1 (left) and of Assumption 2 (right) for a hexagonal cell  $T$  cut by the interface  $\Gamma$ .

**Lemma 3** (Multiplicative trace inequality). *There are real numbers  $c_{\text{mtr}} > 0$  and  $\theta_{\text{mtr}} \geq 1$ , depending on the mesh regularity parameter  $\rho \in (0, 1)$  and the interface regularity parameter  $\gamma \in (0, 1)$ , such that, for all  $T \in \mathcal{T}_h^\Gamma$ , there is  $\tilde{\mathbf{x}}_T \in T$  so that, for all  $i \in \{1, 2\}$  and all  $v \in H^1(T^\dagger)$  with  $T^\dagger = B(\tilde{\mathbf{x}}_T, \theta_{\text{mtr}} h_T)$ ,*

$$\|v\|_{L^2(\partial T^i)} \leq c_{\text{mtr}} \left( h_T^{-\frac{1}{2}} \|v\|_{L^2(T^\dagger)} + \|v\|_{L^2(T^\dagger)}^{\frac{1}{2}} \|\nabla v\|_{L^2(T^\dagger)}^{\frac{1}{2}} \right). \quad (7)$$

*Proof.* The proof is inspired by the ideas from, e.g., [31, Section 6]. Let  $T \in \mathcal{T}_h^\Gamma$  and  $i \in \{1, 2\}$ , and recall that  $\partial T^i = (\partial T)^i \cup T^\Gamma$ . We prove (7) for  $v \in C^1(T^\dagger)$  and then extend this bound to  $H^1(T^\dagger)$  by a density argument. Let us first bound  $\|v\|_{L^2(T^\Gamma)}$ . Integrating, for all  $\mathbf{s} \in T^\Gamma$ , along the segment  $\{\mathbf{p}(\mathbf{s}, t) := (1-t)\hat{\mathbf{x}}_T + t\mathbf{s}, \forall t \in [0, 1]\}$ , where the point  $\hat{\mathbf{x}}_T$  is given by Assumption 1, we obtain

$$v^2(\mathbf{s}) = \int_0^1 \frac{\partial}{\partial t} \left( t^d v(\mathbf{p}(\mathbf{s}, t))^2 \right) dt, \quad \forall \mathbf{s} \in T^\Gamma.$$

Integrating over  $\mathbf{s} \in T^\Gamma$  and developing the derivative with respect to  $t$ , we infer that

$$\|v\|_{L^2(T^\Gamma)}^2 = \int_{T^\Gamma} \int_0^1 (dt^{d-1} v(\mathbf{p}(\mathbf{s}, t))^2 + t^d v(\mathbf{p}(\mathbf{s}, t)) \nabla v(\mathbf{p}(\mathbf{s}, t)) \cdot (\mathbf{s} - \hat{\mathbf{x}}_T)) dt d\mathbf{s}.$$

Let us introduce the cone  $C(T) = \{\mathbf{p}(\mathbf{s}, t), \forall t \in [0, 1], \forall \mathbf{s} \in T^\Gamma\}$ . Since  $d(\hat{\mathbf{x}}_T, T_s\Gamma) \geq \gamma h_T$  (see Assumption 1), the change of variable  $d(\hat{\mathbf{x}}_T, T_s\Gamma) t^{d-1} dt d\mathbf{s} = d\mathbf{p}$  is legitimate. We then obtain that

$$\|v\|_{L^2(T^\Gamma)}^2 = \int_{C(T)} (dv(\mathbf{p}(\mathbf{s}, t))^2 + tv(\mathbf{p}(\mathbf{s}, t)) \nabla v(\mathbf{p}(\mathbf{s}, t)) \cdot (\mathbf{s} - \hat{\mathbf{x}}_T)) d(\hat{\mathbf{x}}_T, T_s\Gamma)^{-1} d\mathbf{p}.$$

Since Assumption 1 implies that  $C(T) \subset T_0^\dagger := B(\hat{\mathbf{x}}_T, \gamma^{-1} h_T)$ , we conclude that

$$\|v\|_{L^2(T^\Gamma)}^2 \leq c_0 \left( h_T^{-1} \|v\|_{L^2(T_0^\dagger)}^2 + \|v\|_{L^2(T_0^\dagger)} \|\nabla v\|_{L^2(T_0^\dagger)} \right),$$

where  $c_0$  depends on the interface regularity parameter  $\gamma \in (0, 1)$ . Let us now bound  $\|v\|_{L^2((\partial T)^i)}$ . Proceeding as in [14, Lemma 1.49] using mesh regularity, we infer that there is a point  $\tilde{\mathbf{x}}_T \in T$  and positive real numbers  $c_1, \theta_1$  depending on the mesh-regularity parameter  $\rho \in (0, 1)$  so that

$$\|v\|_{L^2(\partial T)}^2 \leq c_1 \left( h_T^{-1} \|v\|_{L^2(T_1^\dagger)}^2 + \|v\|_{L^2(T_1^\dagger)} \|\nabla v\|_{L^2(T_1^\dagger)} \right),$$

with  $T_1^\dagger = B(\tilde{\mathbf{x}}_T, \theta_1 h_T)$ . To conclude, we combine the two above bounds using that  $T_0^\dagger \cup T_1^\dagger = B(\hat{\mathbf{x}}_T, \gamma^{-1} h_T) \cup B(\tilde{\mathbf{x}}_T, \theta_1 h_T) \subset B(\tilde{\mathbf{x}}_T, \theta_{\text{mtr}} h_T) =: T^\dagger$  with  $h_T^{-1} \|\hat{\mathbf{x}}_T - \tilde{\mathbf{x}}_T\|_{\ell^2} + \max(\gamma^{-1}, \theta_1) \leq 1 + \gamma^{-1} + \max(\gamma^{-1}, \theta_1) =: \theta_{\text{mtr}}$  (since  $\|\hat{\mathbf{x}}_T - \tilde{\mathbf{x}}_T\|_{\ell^2} \leq \|\hat{\mathbf{x}}_T - \mathbf{s}\|_{\ell^2} + \|\mathbf{s} - \tilde{\mathbf{x}}_T\|_{\ell^2} \leq 1 + \gamma^{-1}$  for all  $\mathbf{s} \in T^\Gamma$ ), and we set  $c_{\text{mtr}} = \max(c_0, c_1)^{\frac{1}{2}}$ .  $\square$

**Lemma 4** (Discrete trace inequality). *Let  $l \in \mathbb{N}$ ,  $l \geq 0$ . There is  $c_{\text{dtr}}$ , depending on the polynomial degree  $l$ , the mesh regularity parameter  $\rho \in (0, 1)$ , and the cut parameter  $\delta \in (0, 1)$ , such that, for all  $T \in \mathcal{T}_h^\Gamma$ , all  $i \in \{1, 2\}$ , and all  $v \in \mathbb{P}^l(T^i)$ ,*

$$\|v\|_{L^2(\partial T^i)} \leq c_{\text{dtr}} h_T^{-\frac{1}{2}} \|v\|_{L^2(T^i)}. \quad (8)$$

*Proof.* Let  $T \in \mathcal{T}_h^\Gamma$ . Let  $i \in \{1, 2\}$ , and let  $v \in \mathbb{P}^l(T^i)$ . Since  $\partial T^i \subset B(\tilde{\mathbf{x}}_{T^i}, h_T)$  and  $B(\tilde{\mathbf{x}}_{T^i}, \delta h_T) \subset T^i$  owing to (6), we observe that

$$\begin{aligned} \|v\|_{L^2(\partial T^i)} &\leq |\partial T^i|^{\frac{1}{2}} \|v\|_{L^\infty(\partial T^i)} \leq |\partial T^i|^{\frac{1}{2}} \|v\|_{L^\infty(B(\tilde{\mathbf{x}}_{T^i}, h_T))} \\ &\leq \hat{c} |\partial T^i|^{\frac{1}{2}} |B(\tilde{\mathbf{x}}_{T^i}, \delta h_T)|^{-\frac{1}{2}} \|v\|_{L^2(B(\tilde{\mathbf{x}}_{T^i}, \delta h_T))} \\ &\leq \hat{c}' |\partial T^i|^{\frac{1}{2}} h_T^{-\frac{d}{2}} \|v\|_{L^2(B(\tilde{\mathbf{x}}_{T^i}, \delta h_T))} \leq \hat{c}' |\partial T^i|^{\frac{1}{2}} h_T^{-\frac{d}{2}} \|v\|_{L^2(T^i)}, \end{aligned}$$

where the factor  $\hat{c}$  results from the inverse inequality  $\|\hat{v}\|_{L^\infty(B(\mathbf{0}, 1))} \leq \hat{c} \|\hat{v}\|_{L^2(B(\mathbf{0}, \delta))}$  for all  $\hat{v} \in \mathbb{P}^l(B(\mathbf{0}, 1))$  and the pullback using the bijective affine map from  $B(\tilde{\mathbf{x}}_{T^i}, h_T)$  to  $B(\mathbf{0}, 1)$ . We conclude by observing that  $|\partial T^i| \leq c h_T^{d-1}$  (with  $c$  depending on  $\rho$ ).  $\square$

*Remark 3.1. (Lemma 4)* For conforming finite elements on unfitted meshes, the discrete trace inequality (8) is invoked only on  $T^\Gamma$ . Here, this inequality needs also to be invoked on  $(\partial T)^i$  since the HHO method involves unknowns attached to the mesh faces, see the proofs of Lemma 6 and of Lemma 12 below.

## 4 The unfitted HHO method

In this section, we describe the unfitted HHO method for the interface problem. Let  $k \geq 0$  be the polynomial degree.

### 4.1 Uncut cells

Let  $\mathcal{T}_h^{\setminus \Gamma} := \mathcal{T}_h^1 \cup \mathcal{T}_h^2$  be the collection of the uncut cells. Let  $T \in \mathcal{T}_h^{\setminus \Gamma}$  and set  $\kappa_T = \kappa^i$  if  $T \in \mathcal{T}_h^i$ ,  $i \in \{1, 2\}$ . We define the following local bilinear form for all  $v, w \in H^1(T)$ :

$$a_T(v, w) = \int_T \kappa_T \nabla v \cdot \nabla w. \quad (9)$$

The classical HHO method is defined locally on each uncut cell  $T \in \mathcal{T}_h^{\setminus \Gamma}$  from a pair of local unknowns which consist of one polynomial of order  $(k+1)$  in  $T$  and a piecewise polynomial of order  $k$  on  $\partial T$  (that is, one polynomial of order  $k$  on each face  $F \in \mathcal{F}_{\partial T}$ ). The local unknowns are generically denoted

$$\hat{v}_T = (v_T, v_{\partial T}) \in \mathbb{P}^{k+1}(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T}) =: \hat{\mathcal{X}}_T^{\setminus \Gamma}, \quad (10)$$





Figure 3: Uncut hexagonal cell. Left:  $k = 0$ ; Right:  $k = 1$ . Each dot attached to a geometric entity (face or cell) symbolizes one degree of freedom (not necessarily a pointwise evaluation).

with the piecewise polynomial space  $\mathbb{P}^k(\mathcal{F}_{\partial T}) = \times_{F \in \mathcal{F}_{\partial T}} \mathbb{P}^k(F)$ . The placement of the discrete unknowns for the uncut cells is illustrated in Figure 3.

There are two key ingredients to devise the local HHO bilinear form. The first one is a reconstruction operator. Let  $\hat{v}_T = (v_T, v_{\partial T}) \in \hat{\mathcal{X}}_T^{\setminus \Gamma}$ . Then, we reconstruct a polynomial  $r_T^{k+1}(\hat{v}_T) \in \mathbb{P}^{k+1}(T)$  by requiring that, for all  $z \in \mathbb{P}^{k+1}(T)$ , the following holds true:

$$a_T(r_T^{k+1}(\hat{v}_T), z) = a_T(v_T, z) - \int_{\partial T} \kappa_T \nabla z \cdot \mathbf{n}_T (v_T - v_{\partial T}), \quad (11)$$

where  $\mathbf{n}_T$  is the unit outward-pointing normal to  $T$ . It is readily seen that  $r_T^{k+1}(\hat{v}_T)$  is uniquely defined by (11) up to an additive constant; one way to fix the constant is to prescribe  $\int_T r_T^{k+1}(\hat{v}_T) = \int_T v_T$  (this choice is irrelevant in what follows). The second ingredient is the stabilization bilinear form defined so that, for all  $\hat{v}_T, \hat{w}_T \in \hat{\mathcal{X}}_T^{\setminus \Gamma}$ ,

$$s_T(\hat{v}_T, \hat{w}_T) = \kappa_T h_T^{-1} \int_{\partial T} \Pi_{\partial T}^k(v_T - v_{\partial T})(w_T - w_{\partial T}), \quad (12)$$

where  $\Pi_{\partial T}^k$  denotes the  $L^2$ -orthogonal projector onto the piecewise polynomial space  $\mathbb{P}^k(\mathcal{F}_{\partial T})$ . Finally, the local HHO bilinear and linear forms to be used when assembling the global discrete problem (see Section 4.3) are as follows: For all  $\hat{v}_T, \hat{w}_T \in \hat{\mathcal{X}}_T^{\setminus \Gamma}$ ,

$$\hat{a}_T^{\setminus \Gamma}(\hat{v}_T, \hat{w}_T) = a_T(r_T^{k+1}(\hat{v}_T), r_T^{k+1}(\hat{w}_T)) + s_T(\hat{v}_T, \hat{w}_T), \quad (13a)$$

$$\hat{\ell}_T^{\setminus \Gamma}(\hat{w}_T) = \int_T f w_T. \quad (13b)$$

*Remark 4.1. (Cell unknowns)* In the classical HHO method, there is some flexibility in the choice of the cell unknowns since one can take them to be polynomials of order  $l \in \{k-1, k, k+1\}$ . In the present context, we will need to work with polynomials of order  $(k+1)$  in the cut cells to achieve optimal approximation properties (see Section 5.2); for simplicity, we consider polynomials of order  $(k+1)$  in the uncut cells as well. Taking polynomials of order  $l \in \{k-1, k\}$  in the uncut cells leads to slightly smaller matrices to be inverted when computing the reconstruction operator from (11), but requires a somewhat more involved design of the stabilization operator than in (12) (see [16, 15]).

## 4.2 Cut cells

Let  $T \in \mathcal{T}_h^\Gamma$ . We use capital letters to denote a generic pair  $V = (v^1, v^2) \in H^1(T^1) \times H^1(T^2)$ . We define the following Nitsche-mortaring bilinear form for all  $V, W \in H^s(T^1) \times H^1(T^2)$ ,  $s > \frac{3}{2}$ :

$$n_T(V, W) = \sum_{i \in \{1, 2\}} \int_{T^i} \kappa^i \nabla v^i \cdot \nabla w^i + n_{T\Gamma}(V, W), \quad (14a)$$

$$n_{T\Gamma}(V, W) = - \int_{T\Gamma} (\kappa \nabla v)^1 \cdot \mathbf{n}_\Gamma \llbracket W \rrbracket_\Gamma + (\kappa \nabla w)^1 \cdot \mathbf{n}_\Gamma \llbracket V \rrbracket_\Gamma - \eta \frac{\kappa^1}{h_T} \llbracket V \rrbracket_\Gamma \llbracket W \rrbracket_\Gamma, \quad (14b)$$

where the user-specified parameter  $\eta$  is such that  $\eta \geq 4c_{\text{dtr}}^2$  where  $c_{\text{dtr}}$  results from the discrete trace inequality (8) with polynomial degree  $l = k$ . Note also that the jump-penalty term is weighted by the lowest value of the diffusion coefficient.

We consider a quadruple of discrete HHO unknowns,  $\hat{V}_T = (V_T, V_{\partial T})$ , where both  $V_T$  and  $V_{\partial T}$  are pairs associated with the partition  $\bar{\Omega} = \bar{\Omega}^1 \cup \bar{\Omega}^2$ , so that

$$V_T = (v_{T^1}, v_{T^2}) \in \mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2), \quad (15)$$

and

$$V_{\partial T} = (v_{(\partial T)^1}, v_{(\partial T)^2}) \in \mathbb{P}^k(\mathcal{F}_{(\partial T)^1}) \times \mathbb{P}^k(\mathcal{F}_{(\partial T)^2}), \quad (16)$$

where  $\mathbb{P}^k(\mathcal{F}_{(\partial T)^i}) := \chi_{F \in \mathcal{F}_{(\partial T)^i}} \mathbb{P}^k(F)$  is the piecewise polynomial space of order  $k$  on  $(\partial T)^i$  based on the (sub-)faces in  $\mathcal{F}_{(\partial T)^i}$ . (Recall that, by definition, all the elements  $F$  of  $\mathcal{F}_{(\partial T)^i}$  are subsets of  $(\partial T)^i = \partial T \cap \Omega^i$ .) Note that we do not introduce any discrete unknown on  $T^\Gamma$ . We use the concise notation  $\hat{V}_T \in \hat{\mathcal{X}}_T^\Gamma$  with

$$\hat{\mathcal{X}}_T^\Gamma = \left( \mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2) \right) \times \left( \mathbb{P}^k(\mathcal{F}_{(\partial T)^1}) \times \mathbb{P}^k(\mathcal{F}_{(\partial T)^2}) \right). \quad (17)$$

The placement of the discrete HHO unknowns in the cut cells for the interface problem is illustrated in Figure 4.

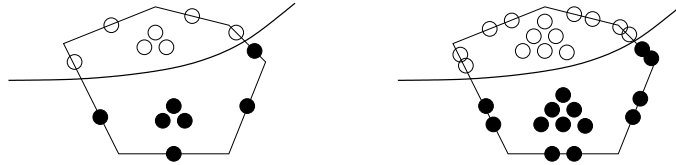


Figure 4: Cut hexagonal cell for the interface problem. The subdomain  $\Omega^1$  is located below  $\Gamma$  with the corresponding HHO unknowns shown by filled circles, and the subdomain  $\Omega^2$  is located above  $\Gamma$  with the corresponding HHO unknowns shown by empty circles. Left:  $k = 0$ ; Right:  $k = 1$ .

As above, there are two key ingredients to devise the local HHO bilinear form: reconstruction and stabilization. Let  $\hat{V}_T \in \hat{\mathcal{X}}_T^\Gamma$ . We reconstruct a pair of polynomials

$R_T^{k+1}(\hat{V}_T) \in \mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2)$  by requiring that, for all  $Z = (z^1, z^2) \in \mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2)$ , the following holds true:

$$n_T(R_T^{k+1}(\hat{V}_T), Z) = n_T(V_T, Z) - \sum_{i \in \{1,2\}} \int_{(\partial T)^i} \kappa^i \nabla z^i \cdot \mathbf{n}_T (v_{T^i} - v_{(\partial T)^i}). \quad (18)$$

It follows from Lemma 5 below that  $R_T^{k+1}(\hat{V}_T)$  is uniquely defined by (18) up to the same additive constant for both of its components; one way to fix the constant is to prescribe  $\sum_{i \in \{1,2\}} \int_{T^i} (R_T^{k+1}(\hat{V}_T))^i = \sum_{i \in \{1,2\}} \int_{T^i} v_{T^i}$  (this choice is irrelevant in what follows). Concerning stabilization, we set for all  $\hat{V}_T, \hat{W}_T \in \hat{\mathcal{X}}_T^\Gamma$ ,

$$s_T(\hat{V}_T, \hat{W}_T) = \sum_{i \in \{1,2\}} \kappa^i h_T^{-1} \int_{(\partial T)^i} \Pi_{(\partial T)^i}^k (v_{T^i} - v_{(\partial T)^i})(w_{T^i} - w_{(\partial T)^i}), \quad (19)$$

where  $\Pi_{(\partial T)^i}^k$  denotes the  $L^2$ -orthogonal projector onto the piecewise polynomial space  $\mathbb{P}^k(\mathcal{F}_{(\partial T)^i})$ . Finally, the local HHO bilinear and linear forms are as follows: For all  $\hat{V}_T, \hat{W}_T \in \hat{\mathcal{X}}_T^\Gamma$ ,

$$\hat{a}_T^\Gamma(\hat{V}_T, \hat{W}_T) = n_T(R_T^{k+1}(\hat{V}_T), R_T^{k+1}(\hat{W}_T)) + s_T(\hat{V}_T, \hat{W}_T), \quad (20a)$$

$$\hat{\ell}_T^\Gamma(\hat{W}_T) = \sum_{i \in \{1,2\}} \int_{T^i} f w_{T^i} + \int_{T^\Gamma} (g_N w_{T^2} + g_D \phi_T(W_T)), \quad (20b)$$

with  $\phi_T(W_T) = -\kappa^1 \nabla w_{T^1} \cdot \mathbf{n}_\Gamma + \eta \kappa^1 h_T^{-1} \llbracket W_T \rrbracket_\Gamma$  (the definition of the integral over  $T^\Gamma$  follows from consistency arguments, see the proof of Lemma 12 below).

### 4.3 The global discrete problem

The mesh faces are collected in the set  $\mathcal{F}_h$  which is partitioned into  $\mathcal{F}_h = \mathcal{F}_h^1 \cup \mathcal{F}_h^\Gamma \cup \mathcal{F}_h^2$ , where  $\mathcal{F}_h^i$ ,  $i \in \{1,2\}$ , collect the mesh faces inside the subdomain  $\Omega^i$  and  $\mathcal{F}_h^\Gamma$  collects the mesh faces cut by the interface. We also define for all  $i \in \{1,2\}$ ,

$$\hat{\mathcal{T}}_h^i := \mathcal{T}_h^i \cup \{T^i = T \cap \Omega^i \mid T \in \mathcal{T}_h^\Gamma\}, \quad (21a)$$

$$\hat{\mathcal{F}}_h^i := \mathcal{F}_h^i \cup \{F^i = F \cap \Omega^i \mid F \in \mathcal{F}_h^\Gamma\}, \quad (21b)$$

i.e.,  $\hat{\mathcal{T}}_h^i$  (resp.,  $\hat{\mathcal{F}}_h^i$ ) is the collection of all the mesh cells (resp., faces) inside  $\Omega^i$  plus the collection of the sub-cells (resp., sub-faces) of the cut cells (resp., cut faces) inside  $\Omega^i$ . Let us set

$$\hat{\mathcal{X}}_h^i := \chi_{T \in \hat{\mathcal{T}}_h^i} \mathbb{P}^{k+1}(T) \times \chi_{F \in \hat{\mathcal{F}}_h^i} \mathbb{P}^k(F). \quad (22)$$

The global discrete space is  $\hat{\mathcal{X}}_h := \hat{\mathcal{X}}_h^1 \times \hat{\mathcal{X}}_h^2$ . Let  $\mathcal{F}_h^\partial$  be the collection of the mesh faces located at the boundary  $\partial\Omega$  (note that the faces in  $\mathcal{F}_h^\partial$  are in one and only one of the subsets  $\mathcal{F}_h^i$ , but not in  $\mathcal{F}_h^\Gamma$  since the interface  $\Gamma$  is located in the interior of  $\Omega$ ). We enforce the homogeneous Dirichlet condition on  $\partial\Omega$  by zeroing out the discrete HHO unknowns attached to the mesh faces in  $\mathcal{F}_h^\partial$ . Let  $i^\partial \in \{1,2\}$  be the index of the subdomain touching the boundary  $\partial\Omega$ . Let  $\hat{\mathcal{X}}_{h0}^{i^\partial}$  be the subspace of  $\hat{\mathcal{X}}_h^{i^\partial}$  composed of all the discrete HHO

unknowns such that their component attached to a mesh face is zero if this face lies on the boundary  $\partial\Omega$ . If  $i^\partial = 1$ , we set  $\hat{\mathcal{X}}_{h0} := \hat{\mathcal{X}}_{h0}^1 \times \hat{\mathcal{X}}_h^2$ ; otherwise, we set  $\hat{\mathcal{X}}_{h0} := \hat{\mathcal{X}}_h^1 \times \hat{\mathcal{X}}_{h0}^2$ .

Let  $\hat{V}_h \in \hat{\mathcal{X}}_{h0}$ . For all  $T \in \mathcal{T}_h^{\setminus\Gamma} = \mathcal{T}_h^1 \cup \mathcal{T}_h^2$ , we denote  $\hat{v}_T = (v_T, v_{(\partial T)}) \in \hat{\mathcal{X}}_T^{\setminus\Gamma}$  (see (10)) the components of  $\hat{V}_h$  attached to the cell  $T$ . For all  $T \in \mathcal{T}_h^\Gamma$ , we denote  $\hat{V}_T = (V_T, V_{\partial T}) \in \hat{\mathcal{X}}_T^\Gamma$  (see (17)) the components of  $\hat{V}_h$  attached to the cell  $T$ . The discrete problem we want to solve reads as follows: Find  $\hat{U}_h \in \hat{\mathcal{X}}_{h0}$  s.t.

$$\hat{a}_h(\hat{U}_h, \hat{W}_h) = \hat{\ell}_h(\hat{W}_h), \quad \forall \hat{W}_h \in \hat{\mathcal{X}}_{h0}, \quad (23)$$

with

$$\hat{a}_h(\hat{V}_h, \hat{W}_h) = \sum_{T \in \mathcal{T}_h^{\setminus\Gamma}} \hat{a}_T^{\setminus\Gamma}(\hat{v}_T, \hat{w}_T) + \sum_{T \in \mathcal{T}_h^\Gamma} \hat{a}_T^\Gamma(\hat{V}_T, \hat{W}_T), \quad (24a)$$

$$\hat{\ell}_h(\hat{W}_h) = \sum_{T \in \mathcal{T}_h^{\setminus\Gamma}} \hat{\ell}_T^{\setminus\Gamma}(\hat{w}_T) + \sum_{T \in \mathcal{T}_h^\Gamma} \hat{\ell}_T^\Gamma(\hat{W}_T), \quad (24b)$$

where  $\hat{a}_T^{\setminus\Gamma}(\cdot, \cdot)$  and  $\hat{\ell}_T^{\setminus\Gamma}(\cdot)$  are defined by (13) for all  $T \in \mathcal{T}_h^{\setminus\Gamma}$  and  $\hat{a}_T^\Gamma(\cdot, \cdot)$  and  $\hat{\ell}_T^\Gamma(\cdot)$  are defined by (20) for all  $T \in \mathcal{T}_h^\Gamma$ .

The discrete problem (23) can be solved efficiently by eliminating locally all the cell unknowns using static condensation. This local elimination leads to a global transmission problem on the mesh skeleton involving only the face unknowns with a stencil that couples unknowns attached to neighboring faces (in the sense of cells). Once this global transmission problem is solved, the cell unknowns are recovered by local solves. We refer the reader, e.g., to [11] for more details in the case of classical HHO methods.

## 5 Analysis

In this section we analyze the convergence of the unfitted HHO method for the interface problem. The proof consists in establishing stability, consistency, and boundedness properties for the discrete forms  $\hat{a}_h$  and  $\hat{\ell}_h$ , and in devising a local approximation operator related to the local reconstruction operators  $r_T^{k+1}$  (see (11)) and  $R_T^{k+1}$  (see (18)). The mesh  $\mathcal{T}_h$  is assumed to satisfy Assumption 1 and Assumption 2 so as to invoke the trace inequalities from Lemma 3 and 4.

In what follows, we often abbreviate  $A \lesssim B$  the inequality  $A \leq CB$  for positive real numbers  $A$  and  $B$ , where the constant  $C$  does not depend on  $\kappa$  nor on the way the interface cuts the mesh-cells, but only depends on the polynomial degree  $k \geq 0$ , the mesh regularity parameter  $\rho \in (0, 1)$ , the interface regularity parameter  $\gamma \in (0, 1)$  from Assumption 1, and the cut parameter  $\delta \in (0, 1)$  from Assumption 2.

### 5.1 Stability and well-posedness

We start with the following stability and boundedness results on the Nitsche-mortaring bilinear form  $n_T$  defined by (14) for all  $T \in \mathcal{T}_h^\Gamma$ . We define the following stability semi-

norm for all  $V = (v^1, v^2) \in H^1(T^1) \times H^1(T^2)$ :

$$|V|_{n_T}^2 := \sum_{i \in \{1,2\}} \kappa^i \|\nabla v^i\|_{T^i}^2 + \eta \frac{\kappa^1}{h_T} \|[[V]]_\Gamma\|_{T^\Gamma}^2. \quad (25)$$

Recall our assumption on the penalty parameter  $\eta \geq 4c_{\text{dtr}}^2$ .

**Lemma 5** (Stability and boundedness of  $n_T$ ). *Let  $T \in \mathcal{T}_h^\Gamma$ . The following holds true for all  $V \in \mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2)$ :*

$$n_T(V, V) \geq \frac{1}{2} |V|_{n_T}^2. \quad (26)$$

Moreover, the following holds true for all  $V, W \in \mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2)$ :

$$|n_T(V, W)| \lesssim |V|_{n_T} |W|_{n_T}, \quad (27)$$

and for all  $V \in H^s(T^1) \times H^1(T^2)$ ,  $s > \frac{3}{2}$ , and all  $W \in \mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2)$ :

$$|n_T(V, W)| \lesssim |V|_{n_T\#} |W|_{n_T}, \quad |V|_{n_T\#}^2 := |V|_{n_T}^2 + \kappa^1 h_T \|\nabla v^1\|_{T^\Gamma}^2. \quad (28)$$

*Proof.* The proof is classical; we sketch it for completeness. Let  $V \in \mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2)$ , and let us set  $\xi = (\sum_{i \in \{1,2\}} \kappa^i \|\nabla v^i\|_{T^i}^2)^{\frac{1}{2}}$  and  $\zeta = (\eta \frac{\kappa^1}{h_T} \|[[V]]_\Gamma\|_{T^\Gamma}^2)^{\frac{1}{2}}$  so that  $|V|_{n_T}^2 = \xi^2 + \zeta^2$ . The definition (14) of  $n_T$  followed by the Cauchy–Schwarz inequality and the discrete trace inequality (8) (applied on  $T^\Gamma$  with  $l = k$ ) yields

$$n_T(V, V) = \xi^2 - 2 \int_{T^\Gamma} (\kappa \nabla v)^1 \cdot \mathbf{n}_\Gamma [[V]]_\Gamma + \zeta^2 \geq \xi^2 - 2c_{\text{dtr}} \eta^{-\frac{1}{2}} \xi \zeta + \zeta^2,$$

so that  $n_T(V, V) \geq \frac{1}{2}(\xi^2 + \zeta^2)$  (i.e., (26)) follows from the assumption that  $\eta \geq 4c_{\text{dtr}}^2$ . Moreover, using the Cauchy–Schwarz inequality, we infer that

$$|n_T(V, W)| \leq |V|_{n_T} |W|_{n_T} + \kappa^1 \|\nabla v^1\|_{T^\Gamma} \|[[W]]_\Gamma\|_{T^\Gamma} + \kappa^1 \|\nabla w^1\|_{T^\Gamma} \|[[V]]_\Gamma\|_{T^\Gamma},$$

so that (27) and (28) follow from the discrete trace inequality (8).  $\square$

We can now address the stability of the local HHO bilinear forms  $\hat{a}_T^\Gamma$  and  $\hat{a}_T^\Gamma$ . For all  $T \in \mathcal{T}_h^{\setminus \Gamma}$ , we consider the local semi-norm used in the analysis of classical HHO methods: For all  $\hat{v}_T = (v_T, v_{\partial T}) \in \hat{\mathcal{X}}_T^\Gamma$ ,

$$|\hat{v}_T|_{\hat{a}_T}^2 := \kappa_T \|\nabla v_T\|_T^2 + \kappa_T h_T^{-1} \|\Pi_{\partial T}^k(v_T - v_{\partial T})\|_{\partial T}^2 = |v_T|_{a_T}^2 + s_T(\hat{v}_T, \hat{v}_T), \quad (29)$$

where we have set  $|v_T|_{a_T}^2 := \kappa_T \|\nabla v_T\|_T^2$ . For all  $T \in \mathcal{T}_h^\Gamma$ , we define the following local semi-norm: For all  $\hat{V}_T = (V_T, V_{\partial T}) = ((v_{T^1}, v_{T^2}), (v_{(\partial T)^1}, v_{(\partial T)^2})) \in \hat{\mathcal{X}}_T^\Gamma$ :

$$\begin{aligned} |\hat{V}_T|_{\hat{a}_T}^2 &:= \sum_{i \in \{1,2\}} \kappa^i \|\nabla v_{T^i}\|_{T^i}^2 + \eta \frac{\kappa^1}{h_T} \|[[V_T]]_\Gamma\|_{T^\Gamma}^2 \\ &\quad + \sum_{i \in \{1,2\}} \kappa^i h_T^{-1} \|\Pi_{(\partial T)^i}^k(v_{T^i} - v_{(\partial T)^i})\|_{(\partial T)^i}^2 = |V_T|_{n_T}^2 + s_T(\hat{V}_T, \hat{V}_T). \end{aligned} \quad (30)$$

**Lemma 6** (Stability). *The following holds true:*

$$\hat{a}_T^{\setminus \Gamma}(\hat{v}_T, \hat{v}_T) \gtrsim |\hat{v}_T|_{\hat{a}_T}^2, \quad \forall T \in \mathcal{T}_h^{\setminus \Gamma}, \forall \hat{v}_T \in \hat{\mathcal{X}}_T^{\setminus \Gamma}, \quad (31a)$$

$$\hat{a}_T^{\Gamma}(\hat{V}_T, \hat{V}_T) \gtrsim |\hat{V}_T|_{\hat{a}_T}^2, \quad \forall T \in \mathcal{T}_h^{\Gamma}, \forall \hat{V}_T \in \hat{\mathcal{X}}_T^{\Gamma}. \quad (31b)$$

*Proof.* The proof of (31a) follows from [16, Lemma 4]. Let us now prove (31b). Let  $T \in \mathcal{T}_h^{\Gamma}$  and let  $\hat{V}_T \in \hat{\mathcal{X}}_T^{\Gamma}$ . Taking  $Z = V_T = (v_{T^1}, v_{T^2})$  in the definition (18) of the reconstruction operator and using the stability of  $n_T$  from Lemma 5, we infer that

$$\begin{aligned} |V_T|_{n_T}^2 &\lesssim n_T(V_T, V_T) \\ &= n_T(R_T^{k+1}(\hat{V}_T), V_T) + \sum_{i \in \{1,2\}} \int_{(\partial T)^i} \kappa^i \nabla v_{T^i} \cdot \mathbf{n}_T (v_{T^i} - v_{(\partial T)^i}). \end{aligned}$$

The first term on the right-hand side is controlled using the boundedness property (27) of  $n_T$  and Young's inequality to hide  $|V_T|_{n_T}$  on the left-hand side. For the second term, we use the Cauchy–Schwarz inequality, the fact that  $(\kappa^i \nabla v_{T^i} \cdot \mathbf{n}_T)|_{(\partial T)^i} \in \mathbb{P}^k(\mathcal{F}_{(\partial T)^i})$ , and the definition (19) of  $s_T(\cdot, \cdot)$  to obtain

$$\int_{(\partial T)^i} \kappa^i \nabla v_{T^i} \cdot \mathbf{n}_T (v_{T^i} - v_{(\partial T)^i}) \leq (\kappa^i)^{\frac{1}{2}} h_T^{\frac{1}{2}} \|\nabla v_{T^i}\|_{(\partial T)^i} s_T(\hat{V}_T, \hat{V}_T)^{\frac{1}{2}}.$$

Then, we invoke the discrete trace inequality (8) on  $(\partial T)^i \subset \partial T^i$  for all  $i \in \{1,2\}$ , and Young's inequality to hide  $(\kappa^i)^{\frac{1}{2}} \|\nabla v_{T^i}\|_{T^i}$  on the left-hand side. Putting everything together, we infer that

$$|V_T|_{n_T}^2 \lesssim |R_T^{k+1}(\hat{V}_T)|_{n_T}^2 + s_T(\hat{V}_T, \hat{V}_T),$$

so that using (30) and the stability of  $n_T$  from Lemma 5, we conclude that

$$\begin{aligned} |\hat{V}_T|_{\hat{a}_T}^2 &= |V_T|_{n_T}^2 + s_T(\hat{V}_T, \hat{V}_T) \\ &\lesssim |R_T^{k+1}(\hat{V}_T)|_{n_T}^2 + s_T(\hat{V}_T, \hat{V}_T) \\ &\lesssim n_T(R_T^{k+1}(\hat{V}_T), R_T^{k+1}(\hat{V}_T)) + s_T(\hat{V}_T, \hat{V}_T) = \hat{a}_T(\hat{V}_T, \hat{V}_T), \end{aligned}$$

which is the expected estimate.  $\square$

Summing the local semi-norms over the mesh cells, we define, for all  $\hat{V}_h \in \hat{\mathcal{X}}_h$ ,

$$|\hat{V}_h|_{\hat{a}_h}^2 := \sum_{T \in \mathcal{T}_h^{\setminus \Gamma}} |\hat{v}_T|_{\hat{a}_T}^2 + \sum_{T \in \mathcal{T}_h^{\Gamma}} |\hat{V}_T|_{\hat{a}_T}^2. \quad (32)$$

Note that  $|\cdot|_{\hat{a}_h}$  defines a norm on the subspace  $\hat{\mathcal{X}}_{h0}$ . Indeed, assume that  $|\hat{V}_h|_{\hat{a}_h} = 0$  for some  $\hat{V}_h \in \hat{\mathcal{X}}_{h0}$ . Then, for all  $T \in \mathcal{T}_h^{\Gamma}$ , we have  $|V_T|_{n_T} = 0$  and  $s_T(\hat{V}_T, \hat{V}_T) = 0$ . The nullity of the first term implies that  $v_{T^1}$  and  $v_{T^2}$  are constant functions that take the same value, and the nullity of the second term implies that  $v_{(\partial T)^1}$  and  $v_{(\partial T)^2}$  are also constant functions that take the same value as  $v_{T^1}$  and  $v_{T^2}$ . Moreover, for all  $T \in \mathcal{T}_h^{\setminus \Gamma}$ ,  $|\hat{v}_T|_{\hat{a}_T} = 0$  implies that  $v_T$  and  $v_{\partial T}$  take the same constant value. We can then propagate this constant value up to the boundary  $\partial\Omega$  where the components of  $\hat{V}_h$  attached to the boundary faces vanish. Thus, all the components of  $\hat{V}_h$  are zero.

**Corollary 7** (Well-posedness). *The discrete problem (23) is well-posed.*

*Proof.* We apply the Lax–Milgram Lemma.  $\square$

## 5.2 Approximation

Let  $u$  be the exact solution with  $u^i := u|_{\Omega^i}$ , for all  $i \in \{1, 2\}$ . We set  $U^{\text{ex}} = (u^1, u^2) \in H^1(\Omega^1) \times H^1(\Omega^2)$ .

### 5.2.1 Uncut cells

Let  $T \in \mathcal{T}_h^{\setminus \Gamma}$ . We set  $u_T^{\text{ex}} = u|_T$  where  $i \in \{1, 2\}$  is s.t.  $T \in \mathcal{T}_h^i$  and we consider the approximation of  $u_T^{\text{ex}}$  in  $T$  defined by

$$j_T^{k+1}(u_T^{\text{ex}}) = \Pi_T^{k+1}(u_T^{\text{ex}}), \quad (33)$$

where  $\Pi_T^{k+1}$  stands for the  $L^2$ -orthogonal projector onto  $\mathbb{P}^{k+1}(T)$  (we use a specific notation  $j_T^{k+1}$  for similarity with cut cells, see below). We introduce the following local norm: For all  $v \in H^s(T)$ ,  $s > \frac{3}{2}$ :

$$\|v\|_{*T}^2 = \kappa_T (\|\nabla v\|_T^2 + h_T \|\nabla v\|_{\partial T}^2 + h_T^{-1} \|v\|_{\partial T}^2). \quad (34)$$

**Lemma 8** (Approximation by  $j_T^{k+1}$ ). *Assume  $U^{\text{ex}} \in H^{k+2}(\Omega^1) \times H^{k+2}(\Omega^2)$ . The following holds true for all  $T \in \mathcal{T}_h^{\setminus \Gamma}$ :*

$$\|j_T^{k+1}(u_T^{\text{ex}}) - u_T^{\text{ex}}\|_{*T} \lesssim \kappa_T^{\frac{1}{2}} h_T^{k+1} |u_T^{\text{ex}}|_{H^{k+2}(T)}. \quad (35)$$

*Proof.* The approximation properties of the  $L^2$ -orthogonal projector are classical on meshes where all the cells can be mapped to a reference cell, see, e.g., [17]. On meshes with polyhedral cells which can be split into a finite number of shape-regular simplices, one can proceed as in the proof of [18, Lem. 5.4] by combining the Poincaré–Steklov inequality in each sub-simplex and the multiplicative trace inequality.  $\square$

Let us now define

$$p_T^{k+1}(u_T^{\text{ex}}) = r_T^{k+1}(j_T^{k+1}(u_T^{\text{ex}})) \in \mathbb{P}^{k+1}(T), \quad (36)$$

where  $r_T^{k+1}$  is the reconstruction operator defined by (11) and

$$\hat{j}_T^{k+1}(u_T^{\text{ex}}) = (j_T^{k+1}(u_T^{\text{ex}}), \Pi_{\partial T}^k(u_T^{\text{ex}})) = (\Pi_T^{k+1}(u_T^{\text{ex}}), \Pi_{\partial T}^k(u_T^{\text{ex}})) \in \hat{\mathcal{X}}_T^{\setminus \Gamma}, \quad (37)$$

where  $\Pi_{\partial T}^k$  stands for the  $L^2$ -orthogonal projector onto the piecewise polynomial space  $\mathbb{P}^k(\mathcal{F}_{\partial T})$ .

**Lemma 9** (Approximation). *Assume that  $U^{\text{ex}} \in H^s(\Omega^1) \times H^s(\Omega^2)$ ,  $s > \frac{3}{2}$ . The following holds true for all  $T \in \mathcal{T}_h^{\setminus \Gamma}$ :*

$$|p_T^{k+1}(u_T^{\text{ex}}) - u_T^{\text{ex}}|_{a_T} + s_T(\hat{j}_T^{k+1}(u_T^{\text{ex}}), \hat{j}_T^{k+1}(u_T^{\text{ex}}))^{\frac{1}{2}} \lesssim \|j_T^{k+1}(u_T^{\text{ex}}) - u_T^{\text{ex}}\|_{*T}. \quad (38)$$

*Proof.* It is shown in [16, Lemma 3] that  $p_T^{k+1}(u_T^{\text{ex}})$  is the elliptic projector of  $u_T^{\text{ex}}$  onto  $\mathbb{P}^{k+1}(T)$ , so that  $a_T(p_T^{k+1}(u_T^{\text{ex}}) - u_T^{\text{ex}}, w) = 0$  for all  $w \in \mathbb{P}^{k+1}(T)$ , and  $|p_T^{k+1}(u_T^{\text{ex}}) - u_T^{\text{ex}}|_{a_T} \leq |j_T^{k+1}(u_T^{\text{ex}}) - u_T^{\text{ex}}|_{a_T}$ . The bound on  $|p_T^{k+1}(u_T^{\text{ex}}) - u_T^{\text{ex}}|_{a_T}$  then follows from  $|\cdot|_{a_T} \leq \|\cdot\|_{*T}$ . To bound  $s_T(\hat{j}_T^{k+1}(u_T^{\text{ex}}), \hat{j}_T^{k+1}(u_T^{\text{ex}}))$ , we proceed as in the proof of (47) below.  $\square$

### 5.2.2 Cut cells

For all  $T \in \mathcal{T}_h^\Gamma$ , let us define the pair

$$U_T^{\text{ex}} = (u_{|T^1}^1, u_{|T^2}^2) \in H^1(T^1) \times H^1(T^2). \quad (39)$$

Let  $E^i : H^1(\Omega^i) \rightarrow H^1(\mathbb{R}^d)$ , for all  $i \in \{1, 2\}$ , be stable extension operators. Recall the ball  $T^\dagger$  introduced in Lemma 3 and observe that  $T \subset T^\dagger$  since  $\theta_{\text{mtr}} \geq 1$ . We construct an approximation of the pair  $U_T^{\text{ex}}$  in  $T$  by setting

$$J_T^{k+1}(U^{\text{ex}}) := (\Pi_{T^\dagger}^{k+1}(E^1(u^1))_{|T^1}, \Pi_{T^\dagger}^{k+1}(E^2(u^2))_{|T^2}) \in \mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2), \quad (40)$$

where  $\Pi_{T^\dagger}^{k+1}$  stands for the  $L^2$ -orthogonal projector onto  $\mathbb{P}^{k+1}(T^\dagger)$  (we do not project using the set  $T^i$ , but the larger set  $T^\dagger$ , to avoid dealing with approximation properties on  $T^i$ ). We introduce the following local norm: For all  $V = (v^1, v^2) \in H^s(T^1) \times H^s(T^2)$ ,  $s > \frac{3}{2}$ ,

$$\begin{aligned} \|V\|_{*T}^2 = & \sum_{i \in \{1, 2\}} \kappa^i (\|\nabla v^i\|_{T^i}^2 + h_T \|\nabla v^i\|_{(\partial T)^i}^2 + h_T^{-1} \|v^i\|_{(\partial T)^i}^2) \\ & + \kappa^1 (h_T \|\nabla v^1\|_{T^\Gamma}^2 + h_T^{-1} \|\llbracket V \rrbracket_\Gamma\|_{T^\Gamma}^2) + \kappa^2 h_T \|\nabla v^2\|_{T^\Gamma}^2. \end{aligned} \quad (41)$$

Note that  $|V|_{n_T} \leq |V|_{n_T^\#} \leq \|V\|_{*T}$ .

**Lemma 10** (Approximation by  $J_T^{k+1}$ ). *Assume  $U^{\text{ex}} \in H^{k+2}(\Omega^1) \times H^{k+2}(\Omega^2)$ . The following holds true for all  $T \in \mathcal{T}_h^\Gamma$ :*

$$\|J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}\|_{*T} \lesssim \sum_{i \in \{1, 2\}} (\kappa^i)^{\frac{1}{2}} h_T^{k+1} |E^i(u^i)|_{H^{k+2}(T^\dagger)}. \quad (42)$$

*Proof.* We need to bound the six terms on the right-hand side of (41). The bound on the norm on  $T^i$  is straightforward since this norm can be bounded by the norm on  $T^\dagger$  where we can use the classical approximation properties of  $\Pi_{T^\dagger}^{k+1}$  (recall that  $T \subset T^\dagger$ ). To bound the three norms on  $(\partial T)^i$  and the two norms on  $T^\Gamma$ , we use the multiplicative trace inequality from Lemma 3 and the approximation properties of  $\Pi_{T^\dagger}^{k+1}$  on  $T^\dagger$ .  $\square$

Let us now define

$$P_T^{k+1}(U^{\text{ex}}) = R_T^{k+1}(\hat{J}_T^{k+1}(U^{\text{ex}})) \in \mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2), \quad (43)$$

where  $R_T^{k+1}$  is the reconstruction operator defined by (18) and

$$\hat{J}_T^{k+1}(U^{\text{ex}}) := (J_T^{k+1}(U^{\text{ex}}), (\Pi_{(\partial T)^1}^k(u^1), \Pi_{(\partial T)^2}^k(u^2))) \in \hat{\mathcal{X}}_T^\Gamma, \quad (44)$$

so that  $\hat{J}_T^{k+1}(U^{\text{ex}}) = ((\Pi_{T^\dagger}^{k+1}(E^1(u^1))_{|T^1}, \Pi_{T^\dagger}^{k+1}(E^2(u^2))_{|T^2}), (\Pi_{(\partial T)^1}^k(u^1), \Pi_{(\partial T)^2}^k(u^2)))$ .

**Lemma 11** (Approximation). *Assume that  $U^{\text{ex}} \in H^s(\Omega^1) \times H^s(\Omega^2)$ ,  $s > \frac{3}{2}$ . The following holds true for all  $T \in \mathcal{T}_h^\Gamma$ : (i) For all  $W_T \in \mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2)$ ,*

$$n_T(P_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}, W_T) \lesssim \|J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}\|_{*T} |W_T|_{n_T}. \quad (45)$$

(ii) *We have*

$$|P_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}|_{n_T} \lesssim \|J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}\|_{*T}. \quad (46)$$

(iii) *We have*

$$s_T(\hat{J}_T^{k+1}(U^{\text{ex}}), \hat{J}_T^{k+1}(U^{\text{ex}}))^{\frac{1}{2}} \lesssim \|J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}\|_{*T}. \quad (47)$$



*Proof.* Let us first prove (45). Let  $W_T \in \mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2)$ . Using the definition (18) of the reconstruction operator, we infer that

$$\begin{aligned} n_T(P_T^{k+1}(U^{\text{ex}}), W_T) &= n_T(R_T^{k+1}(\hat{J}_T^{k+1}(U^{\text{ex}})), W_T) \\ &= n_T(J_T^{k+1}(U^{\text{ex}}), W_T) \\ &\quad - \sum_{i \in \{1,2\}} \int_{(\partial T)^i} \kappa^i \nabla w_T^i \cdot \mathbf{n}_T ((J_T^{k+1}(U^{\text{ex}}))^i - \Pi_{(\partial T)^i}^k(u^i)) \\ &= n_T(J_T^{k+1}(U^{\text{ex}}), W_T) \\ &\quad - \sum_{i \in \{1,2\}} \int_{(\partial T)^i} \kappa^i \nabla w_T^i \cdot \mathbf{n}_T ((J_T^{k+1}(U^{\text{ex}}))^i - u^i), \end{aligned}$$

where we have exploited the choice for the face polynomials in the definition (44) of  $\hat{J}_T^{k+1}(U^{\text{ex}})$  and the fact that  $\kappa^i \nabla w_T^i \cdot \mathbf{n}_T \in \mathbb{P}^k(\mathcal{F}_{(\partial T)^i})$ . Since  $(U_T^{\text{ex}})_{|(\partial T)^i}^i = u_{|(\partial T)^i}^i$ , we obtain

$$\begin{aligned} n_T(P_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}, W_T) &= n_T(J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}, W_T) \\ &\quad - \sum_{i \in \{1,2\}} \int_{(\partial T)^i} \kappa^i \nabla w_T^i \cdot \mathbf{n}_T (J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}})^i. \end{aligned}$$

To bound the first term on the right-hand side, we use the boundedness property (28) of  $n_T(\cdot, \cdot)$  from Lemma 5 and  $|\cdot|_{n_T} \leq \|\cdot\|_{*T}$ . To bound the second term, we use the Cauchy–Schwarz inequality followed by the discrete trace inequality (8) to bound  $\|\nabla w_T^i\|_{(\partial T)^i}$ .

Let us now prove (46). Let us set  $Z_T = P_T^{k+1}(U^{\text{ex}}) - J_T^{k+1}(U^{\text{ex}}) \in \mathbb{P}^{k+1}(T^1) \times \mathbb{P}^{k+1}(T^2)$ . Using the stability of  $n_T$  from Lemma 5, we have

$$\begin{aligned} |Z_T|_{n_T}^2 &\lesssim n_T(Z_T, Z_T) \\ &= n_T(P_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}, Z_T) + n_T(U_T^{\text{ex}} - J_T^{k+1}(U^{\text{ex}}), Z_T). \end{aligned}$$

Using (45), we can estimate the first term on the right-hand side as follows:

$$n_T(P_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}, Z_T) \lesssim \|J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}\|_{*T} |Z_T|_{n_T}.$$

Concerning the second term, we invoke the boundedness property (28) of  $n_T(\cdot, \cdot)$  from Lemma 5 and  $|\cdot|_{n_T} \leq \|\cdot\|_{*T}$  to infer that

$$\begin{aligned} n_T(U_T^{\text{ex}} - J_T^{k+1}(U^{\text{ex}}), Z_T) &\lesssim |J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}|_{n_T} |Z_T|_{n_T} \\ &\leq \|J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}\|_{*T} |Z_T|_{n_T}. \end{aligned}$$

Combining these two bounds, we infer that

$$|Z_T|_{n_T} \lesssim \|J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}\|_{*T}.$$

Finally, using a triangle inequality leads to

$$|P_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}|_{n_T} \leq |Z_T|_{n_T} + |J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}|_{n_T},$$

which leads to the expected estimate since  $|\cdot|_{n_T} \leq \|\cdot\|_{*T}$ .

Finally, let us prove (47). We have

$$s_T(\hat{J}_T^{k+1}(U^{\text{ex}}), \hat{J}_T^{k+1}(U^{\text{ex}})) = \sum_{i \in \{1,2\}} \kappa^i h_T^{-1} \|\Pi_{(\partial T)^i}^k((J_T^{k+1}(U^{\text{ex}}))^i - \Pi_{(\partial T)^i}^k(u^i))\|_{(\partial T)^i}^2,$$

and observing that

$$\begin{aligned} \|\Pi_{(\partial T)^i}^k((J_T^{k+1}(U^{\text{ex}}))^i - \Pi_{(\partial T)^i}^k(u^i))\|_{(\partial T)^i} &= \|\Pi_{(\partial T)^i}^k((J_T^{k+1}(U^{\text{ex}}))^i - u^i)\|_{(\partial T)^i} \\ &= \|\Pi_{(\partial T)^i}^k((J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}})^i)\|_{(\partial T)^i} \\ &\leq \|(J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}})^i\|_{(\partial T)^i}, \end{aligned}$$

we infer the expected estimate.  $\square$

### 5.3 Consistency and boundedness

We can now derive our key estimate regarding the consistency of the discrete problem (23).

**Lemma 12** (Consistency and boundedness). *Assume that  $U^{\text{ex}} \in H^s(\Omega^1) \times H^s(\Omega^2)$ ,  $s > \frac{3}{2}$ . Let  $\hat{U}_h \in \hat{\mathcal{X}}_{h0}$  solve (23). For all  $\hat{W}_h \in \hat{\mathcal{X}}_{h0}$ , let us define*

$$\mathcal{F}(\hat{W}_h) = \sum_{T \in \mathcal{T}_h^{\setminus \Gamma}} \hat{a}_T^{\setminus \Gamma}(\hat{J}_T^{k+1}(u_T^{\text{ex}}) - \hat{u}_T, \hat{w}_T) + \sum_{T \in \mathcal{T}_h^{\Gamma}} \hat{a}_T^{\Gamma}(\hat{J}_T^{k+1}(U^{\text{ex}}) - \hat{U}_T, \hat{W}_T).$$

Recall that  $|\cdot|_{\hat{a}_h}$  is defined by (32). The following holds true:

$$|\mathcal{F}(\hat{W}_h)| \lesssim \left( \sum_{T \in \mathcal{T}_h^{\setminus \Gamma}} \|\hat{J}_T^{k+1}(u_T^{\text{ex}}) - u_T^{\text{ex}}\|_{*T}^2 + \sum_{T \in \mathcal{T}_h^{\Gamma}} \|J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}\|_{*T}^2 \right)^{\frac{1}{2}} |\hat{W}_h|_{\hat{a}_h}.$$

*Proof.* We first observe that, for all  $T \in \mathcal{T}_h^{\setminus \Gamma}$ ,

$$\begin{aligned} \hat{a}_T^{\setminus \Gamma}(\hat{J}_T^{k+1}(u_T^{\text{ex}}), \hat{w}_T) &= a_T(p_T^{k+1}(u_T^{\text{ex}}), r_T^{k+1}(\hat{w}_T)) + s_T(\hat{J}_T^{k+1}(u_T^{\text{ex}}), \hat{w}_T) \\ &= a_T(p_T^{k+1}(u_T^{\text{ex}}), w_T) + s_T(\hat{J}_T^{k+1}(u_T^{\text{ex}}), \hat{w}_T) \\ &\quad - \int_{\partial T} \kappa_T \nabla p_T^{k+1}(u_T^{\text{ex}}) \cdot \mathbf{n}_T (w_T - w_{\partial T}), \end{aligned}$$

and for all  $T \in \mathcal{T}_h^{\Gamma}$ ,

$$\begin{aligned} \hat{a}_T^{\Gamma}(\hat{J}_T^{k+1}(U^{\text{ex}}), \hat{W}_T) &= n_T(P_T^{k+1}(U^{\text{ex}}), R_T^{k+1}(\hat{W}_T)) + s_T(\hat{J}_T^{k+1}(U^{\text{ex}}), \hat{W}_T) \\ &= n_T(P_T^{k+1}(U^{\text{ex}}), W_T) + s_T(\hat{J}_T^{k+1}(U^{\text{ex}}), \hat{W}_T) \\ &\quad - \sum_{i \in \{1,2\}} \int_{(\partial T)^i} (\kappa \nabla P_T^{k+1}(U^{\text{ex}}))^i \cdot \mathbf{n}_T (w_{T^i} - w_{(\partial T)^i}), \end{aligned}$$

where we have used the definitions (36) and (43) of  $p_T^{k+1}$  and  $P_T^{k+1}$  and the definitions (11) and (18) of the reconstruction operators for  $r_T^{k+1}(\hat{w}_T)$  and  $R_T^{k+1}(\hat{W}_T)$  (and the symmetry

of the bilinear forms  $a_T$  and  $n_T$ ). Moreover, using the fact that the discrete solution solves (23), we infer that

$$\sum_{T \in \mathcal{T}_h^{\setminus \Gamma}} \hat{a}_T^{\setminus \Gamma}(\hat{u}_T, \hat{w}_T) + \sum_{T \in \mathcal{T}_h^{\Gamma}} \hat{a}_T^{\Gamma}(\hat{U}_T, \hat{W}_T) =: \Psi^{\setminus \Gamma} + \Psi^{\Gamma},$$

where

$$\Psi^{\setminus \Gamma} = \sum_{T \in \mathcal{T}_h^{\setminus \Gamma}} \int_T f w_T = \sum_{T \in \mathcal{T}_h^{\setminus \Gamma}} \left( \int_T \kappa_T \nabla u_T^{\text{ex}} \cdot \nabla w_T - \int_{\partial T} \kappa_T \nabla u_T^{\text{ex}} \cdot \mathbf{n}_T w_T \right),$$

and

$$\begin{aligned} \Psi^{\Gamma} &= \sum_{T \in \mathcal{T}_h^{\Gamma}} \left( \sum_{i \in \{1,2\}} \int_{T^i} f w_{T^i} + \int_{T^{\Gamma}} (g_N w_{T^2} + g_D \phi_T(W_T)) \right) \\ &= \sum_{T \in \mathcal{T}_h^{\Gamma}} \left( \sum_{i \in \{1,2\}} \int_{T^i} -\nabla \cdot (\kappa^i \nabla u^i) w_{T^i} + \int_{T^{\Gamma}} (g_N w_{T^2} + g_D \phi_T(W_T)) \right) \\ &= \sum_{T \in \mathcal{T}_h^{\Gamma}} \left( \sum_{i \in \{1,2\}} \left( \int_{T^i} \kappa^i \nabla u^i \cdot \nabla w_{T^i} - \int_{(\partial T)^i} (\kappa \nabla u)^i \cdot \mathbf{n}_T w_{T^i} \right) + n_{T\Gamma}(U_T^{\text{ex}}, W_T) \right), \end{aligned}$$

where we have used the following identity:

$$\begin{aligned} & - \sum_{i \in \{1,2\}} \int_{T^{\Gamma}} \kappa^i \nabla u^i \cdot \mathbf{n}_{T^i} w_{T^i} + \int_{T^{\Gamma}} (g_N w_{T^2} + g_D \phi_T(W_T)) \\ &= - \int_{T^{\Gamma}} \kappa^1 \nabla u^1 \cdot \mathbf{n}_{\Gamma} \llbracket W_T \rrbracket_{\Gamma} + \int_{T^{\Gamma}} g_D \phi_T(W_T) = n_{T\Gamma}(U_T^{\text{ex}}, W_T), \end{aligned}$$

recalling that  $\llbracket \kappa \nabla u \rrbracket_{\Gamma} \cdot \mathbf{n}_{\Gamma} = g_N$  and  $\llbracket u \rrbracket_{\Gamma} = g_D$ . Therefore, we have

$$\Psi^{\Gamma} = \sum_{T \in \mathcal{T}_h^{\Gamma}} \left( n_{T\Gamma}(U_T^{\text{ex}}, W_T) - \sum_{i \in \{1,2\}} \int_{(\partial T)^i} (\kappa \nabla U_T^{\text{ex}})^i \cdot \mathbf{n}_T w_{T^i} \right).$$

Putting the above identities together leads to  $\mathcal{F}(\hat{W}_h) = \mathcal{F}^{\setminus \Gamma}(\hat{W}_h) + \mathcal{F}^{\Gamma}(\hat{W}_h)$  with

$$\begin{aligned} \mathcal{F}^{\setminus \Gamma}(\hat{W}_h) &= \sum_{T \in \mathcal{T}_h^{\setminus \Gamma}} \left( a_T(p_T^{k+1}(u_T^{\text{ex}}) - u_T^{\text{ex}}, w_T) + s_T(\hat{J}_T^{k+1}(u_T^{\text{ex}}), \hat{w}_T) \right. \\ &\quad \left. - \int_{\partial T} \kappa_T \nabla(p_T^{k+1}(u_T^{\text{ex}}) - u_T^{\text{ex}}) \cdot \mathbf{n}_T (w_T - w_{\partial T}) \right), \\ \mathcal{F}^{\Gamma}(\hat{W}_h) &= \sum_{T \in \mathcal{T}_h^{\Gamma}} \left( n_T(P_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}, W_T) + s_T(\hat{J}_T^{k+1}(U^{\text{ex}}), \hat{W}_T) \right. \\ &\quad \left. - \sum_{i \in \{1,2\}} \int_{(\partial T)^i} (\kappa \nabla(P_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}))^i \cdot \mathbf{n}_T (w_{T^i} - w_{(\partial T)^i}) \right). \end{aligned}$$

where we have used the continuity of the exact fluxes across  $\partial T$  for all  $T \in \mathcal{T}_h^{\setminus \Gamma}$  and across  $(\partial T)^i$  for all  $i \in \{1, 2\}$  and  $T \in \mathcal{T}_h^\Gamma$  to add/subtract  $w_{\partial T}$  and  $w_{(\partial T)^i}$  in the integrals over  $\partial T$  and  $(\partial T)^i$ , respectively. It remains to bound the three terms composing  $\mathcal{F}^\Gamma(\hat{W}_h)$  and  $\mathcal{F}^\Gamma(\hat{W}_h)$  using Lemma 9 and Lemma 11, respectively. We only detail the bound on the three terms composing  $\mathcal{F}^\Gamma(\hat{W}_h)$  since the bound on  $\mathcal{F}^{\setminus \Gamma}(\hat{W}_h)$  uses similar arguments. To bound the first term, we use (45) and to bound the second term, we use (47). For the third term, we use the Cauchy–Schwarz inequality so that we need to bound  $(\kappa^i)^{\frac{1}{2}} h_T^{\frac{1}{2}} \|\nabla(P_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}})^i\|_{(\partial T)^i}$  for all  $i \in \{1, 2\}$ . We can then add/subtract  $(J_T^{k+1}(U^{\text{ex}}))^i$  and use the triangle inequality to obtain

$$\begin{aligned} (\kappa^i)^{\frac{1}{2}} h_T^{\frac{1}{2}} \|\nabla(P_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}})^i\|_{(\partial T)^i} &\leq (\kappa^i)^{\frac{1}{2}} h_T^{\frac{1}{2}} \|\nabla(P_T^{k+1}(U^{\text{ex}}) - J_T^{k+1}(U^{\text{ex}}))^i\|_{(\partial T)^i} \\ &\quad + (\kappa^i)^{\frac{1}{2}} h_T^{\frac{1}{2}} \|\nabla(J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}})^i\|_{(\partial T)^i}. \end{aligned}$$

Since the second term on the right-hand side is bounded by  $\|J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}\|_{*T}$ , we can focus on the first term. Using the discrete trace inequality (8) followed by the triangle inequality where we add/subtract  $(U_T^{\text{ex}})^i$ , we infer that

$$\begin{aligned} (\kappa^i)^{\frac{1}{2}} h_T^{\frac{1}{2}} \|\nabla(P_T^{k+1}(U^{\text{ex}}) - J_T^{k+1}(U^{\text{ex}}))^i\|_{(\partial T)^i} &\lesssim (\kappa^i)^{\frac{1}{2}} \|\nabla(P_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}})^i\|_{T^i} \\ &\quad + (\kappa^i)^{\frac{1}{2}} \|\nabla(U_T^{\text{ex}} - J_T^{k+1}(U^{\text{ex}}))^i\|_{T^i}. \end{aligned}$$

To conclude, we bound the first term using (46), whereas the second term is readily bounded by  $\|J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}\|_{*T}$ .  $\square$

## 5.4 Main result

We can now state our main result on the error analysis.

**Theorem 13** (Error estimate). *Assume that  $U^{\text{ex}} \in H^s(\Omega^1) \times H^s(\Omega^2)$ ,  $s > \frac{3}{2}$ . Let  $\hat{U}_h \in \hat{\mathcal{X}}_{h0}$  solve (23). Then, the following bound holds true:*

$$\begin{aligned} \mathcal{E} &:= \sum_{T \in \mathcal{T}_h^{\setminus \Gamma}} \kappa_T \|\nabla(u_T^{\text{ex}} - u_T)\|_T^2 + \sum_{T \in \mathcal{T}_h^\Gamma} \sum_{i \in \{1, 2\}} \kappa^i \|\nabla(U_T^{\text{ex}} - U_T)^i\|_{T^i}^2 \\ &\quad + \sum_{T \in \mathcal{T}_h^\Gamma} \kappa^1 h_T^{-1} \|g_D - \llbracket U_T \rrbracket_\Gamma\|_{T^\Gamma}^2 + \sum_{T \in \mathcal{T}_h^\Gamma} (\kappa^2)^{-1} h_T \|g_N - \llbracket \kappa \nabla U_T \rrbracket_\Gamma \cdot \mathbf{n}_\Gamma\|_{T^\Gamma}^2 \\ &\lesssim \sum_{T \in \mathcal{T}_h^{\setminus \Gamma}} \|j_T^{k+1}(u_T^{\text{ex}}) - u_T^{\text{ex}}\|_{*T}^2 + \sum_{T \in \mathcal{T}_h^\Gamma} \|J_T^{k+1}(U^{\text{ex}}) - U_T^{\text{ex}}\|_{*T}^2 =: \mathcal{B}. \end{aligned} \quad (48)$$

Moreover, if  $U^{\text{ex}} \in H^{k+2}(\Omega^1) \times H^{k+2}(\Omega^2)$ , the following bounds hold true:

$$\begin{aligned} \mathcal{E} &\lesssim \sum_{T \in \mathcal{T}_h^{\setminus \Gamma}} \kappa_T h_T^{2(k+1)} |u_T^{\text{ex}}|_{H^{k+2}(T)}^2 + \sum_{T \in \mathcal{T}_h^\Gamma} \sum_{i \in \{1, 2\}} \kappa^i h_T^{2(k+1)} |E^i(u^i)|_{H^{k+2}(T^\Gamma)}^2 \\ &\lesssim \sum_{i \in \{1, 2\}} \kappa^i h^{2(k+1)} |u^i|_{H^{k+2}(\Omega^i)}^2. \end{aligned} \quad (49)$$

*Proof.* Let  $\hat{J}_h \in \hat{\mathcal{X}}_{h0}$  be such that its local components attached to the cells  $T \in \mathcal{T}_h^{\setminus \Gamma}$  are  $\hat{J}_T := \hat{J}_T^{k+1}(u_T^{\text{ex}})$  and those attached to the cells  $T \in \mathcal{T}_h^\Gamma$  are  $\hat{J}_T := \hat{J}_T^{k+1}(U^{\text{ex}})$  (the face components of  $\hat{J}^h$  are indeed well defined). Using stability (Lemma 6 and (32)), consistency/boundedness (Lemma 12), and the Cauchy–Schwarz inequality, we infer that

$$\begin{aligned} |\hat{J}_h - \hat{U}_h|_{\hat{a}_h}^2 &= \sum_{T \in \mathcal{T}_h^{\setminus \Gamma}} |\hat{J}_T - \hat{u}_T|_{\hat{a}_T}^2 + \sum_{T \in \mathcal{T}_h^\Gamma} |\hat{J}_T - \hat{U}_T|_{\hat{a}_T}^2 \\ &\lesssim \sum_{T \in \mathcal{T}_h^{\setminus \Gamma}} \hat{a}_T^\Gamma (\hat{J}_T - \hat{u}_T, \hat{J}_T - \hat{u}_T) + \sum_{T \in \mathcal{T}_h^\Gamma} \hat{a}_T^\Gamma (\hat{J}_T - \hat{U}_T, \hat{J}_T - \hat{U}_T) \\ &= \mathcal{F}(\hat{J}_h - \hat{U}_h) \lesssim \mathcal{B}^{\frac{1}{2}} |\hat{J}_h - \hat{U}_h|_{\hat{a}_h}. \end{aligned}$$

This implies that

$$\sum_{T \in \mathcal{T}_h^{\setminus \Gamma}} |\hat{J}_T - \hat{u}_T|_{\hat{a}_T}^2 + \sum_{T \in \mathcal{T}_h^\Gamma} |\hat{J}_T - \hat{U}_T|_{\hat{a}_T}^2 \lesssim \mathcal{B}.$$

Recalling the definitions (29) and (30) of  $|\cdot|_{\hat{a}_T}$ , we infer that

$$\sum_{T \in \mathcal{T}_h^{\setminus \Gamma}} |j_T^{k+1}(u_T^{\text{ex}}) - u_T|_{a_T}^2 + \sum_{T \in \mathcal{T}_h^\Gamma} |J_T^{k+1}(U^{\text{ex}}) - U_T|_{n_T}^2 \lesssim \mathcal{B}, \quad (50)$$

and using the discrete trace inequality (8), we also infer that

$$\sum_{i \in \{1,2\}} \kappa^i h_T \|\nabla(J_T^{k+1}(U^{\text{ex}}) - U_T)^i\|_{T^\Gamma}^2 \lesssim \mathcal{B}. \quad (51)$$

For all  $T \in \mathcal{T}_h^{\setminus \Gamma}$ , we add/subtract  $u_T^{\text{ex}}$  in (50) and we use that  $|\cdot|_{a_T} \leq \|\cdot\|_{*T}$ ; for all  $T \in \mathcal{T}_h^\Gamma$ , we add/subtract  $U_T^{\text{ex}}$  in (50)-(51) and we use that  $|\cdot|_{n_T}^2 + \sum_{i \in \{1,2\}} \kappa^i h_T \|\nabla(\cdot)\|_{T^\Gamma}^2 \leq \|\cdot\|_{*T}^2$ ,  $\llbracket U_T^{\text{ex}} \rrbracket_\Gamma = g_D$ , and

$$(\kappa^2)^{-\frac{1}{2}} h_T^{\frac{1}{2}} \|g_N - \llbracket \kappa \nabla U_T \rrbracket_\Gamma \cdot \mathbf{n}_\Gamma\|_{T^\Gamma} \leq \sum_{i \in \{1,2\}} (\kappa^i h_T)^{\frac{1}{2}} \|\nabla(U_T^{\text{ex}} - U_T)^i\|_{T^\Gamma},$$

since  $\llbracket \kappa \nabla U_T^{\text{ex}} \rrbracket_\Gamma \cdot \mathbf{n}_\Gamma = g_N$  and  $\kappa^1 < \kappa^2$ . This leads to (48). Finally, the estimate (49) follows by combining (48) with Lemmas 8 and 10.  $\square$

## 6 Building the mesh

In this section, we show how to build a mesh satisfying Assumption 1 and Assumption 2. Our goal is not to propose an optimized construction, but simply to show that both Assumptions can be satisfied. A more practically-oriented discussion on algorithmic aspects is postponed to future work. We assume that we are initially given a shape-regular (polyhedral) mesh. Our goal is to show that we can satisfy Assumption 1 by refining the mesh and then Assumption 2 by means of a local cell-agglomeration procedure.

The shape-regularity of the mesh sequence implies that there is  $\rho \in (0, 1)$  so that the following geometric properties hold true for all  $T \in \mathcal{T}_h$ , (i) there is  $\mathbf{x}_T \in T$  so that

$B(\mathbf{x}_T, \rho h_T) \subset T$ ; (ii)  $T_\rho := \{\mathbf{x} \in \mathbb{R}^d, d(\mathbf{x}, T) \leq \rho h_T\} \subset \Delta(T)$  where  $\Delta(T) := \{T' \in \mathcal{T}_h \mid T \cap T' \neq \emptyset\}$  is the collection of the mesh cells touching  $T$ ; (iii)  $\rho \max_{T' \in \Delta(T)} h_{T'} \leq \min_{T'' \in \Delta(T)} h_{T''}$ ; (iv) for all  $\mathbf{x} \in \Omega$  with  $d(\mathbf{x}, \partial\Omega) \geq h$ , and all  $\alpha \in (0, 1)$ , letting  $T_1 \in \mathcal{T}_h$  be s.t.  $\mathbf{x} \in T_1$ , there is  $T_2 \in \mathcal{T}_h$  s.t.  $T_2 \cap B(\mathbf{x}, \alpha h_{T_1})$  has positive  $d$ -measure and there is  $\mathbf{x}_{T_2} \in T_2$  so that  $B(\mathbf{x}_{T_2}, \rho \alpha h_{T_2}) \subset T_2 \cap B(\mathbf{x}, \alpha h_{T_1})$ ; this last property means that for any open ball (not too close to the boundary), there is at least one mesh cell s.t. its intersection with this ball contains a smaller ball with equivalent diameter.

## 6.1 Assumption 1: mesh refinement

Let us show that Assumption 1 can be satisfied if the mesh is fine enough.

**Lemma 14** (Assumption 1). *Assumption 1 holds true with  $\gamma = \frac{1}{4}$  provided  $hM \leq 1$  where  $M$  is an upper bound on the curvature of  $\Gamma$ .*

*Proof.* Let  $T \in \mathcal{T}_h^\Gamma$ . Fix a point  $\mathbf{s}_0 \in T^\Gamma$  and introduce the local coordinates  $\boldsymbol{\xi} = (\boldsymbol{\xi}', \xi_d)$ , with zero at  $\mathbf{s}_0$ , where  $\boldsymbol{\xi}' \in \mathbb{R}^{d-1}$  are the coordinates in the tangent plane  $T_{\mathbf{s}_0}\Gamma$  and  $\xi_d$  is the coordinate in the normal direction to the tangent plane at  $\mathbf{s}_0$ . Owing to the assumption  $hM \leq 1$ , we can write  $T^\Gamma = \{\mathbf{s} := (\boldsymbol{\xi}', \psi(\boldsymbol{\xi}')), \boldsymbol{\xi}' \in V(\mathbf{0})\}$  where  $V(\mathbf{0})$  is a neighborhood of  $\mathbf{0}$  in  $\mathbb{R}^{d-1}$  and  $\psi : V(\mathbf{0}) \rightarrow \mathbb{R}$  is a smooth map. Note that  $\psi(\mathbf{0}) = 0$ ,  $\nabla_{\boldsymbol{\xi}'} \psi(\mathbf{0}) = \mathbf{0}$ , and that a normal vector to the tangent plane  $T_s\Gamma$  is  $\mathbf{n}(\boldsymbol{\xi}') = (-\nabla_{\boldsymbol{\xi}'} \psi(\boldsymbol{\xi}'), 1)$ . Let us set  $\hat{\mathbf{x}}_T = (\mathbf{0}, -2h_T)$  and consider the function

$$f(\boldsymbol{\xi}') = (\mathbf{s} - \hat{\mathbf{x}}_T) \cdot \mathbf{n}_\Gamma(\boldsymbol{\xi}') = -\boldsymbol{\xi}' \cdot \nabla_{\boldsymbol{\xi}'} \psi(\boldsymbol{\xi}') + \psi(\boldsymbol{\xi}') + 2h_T.$$

Then  $f(\mathbf{0}) = 2h_T$ , and since  $\nabla_{\boldsymbol{\xi}'} f(\boldsymbol{\xi}') = -\boldsymbol{\xi}' \cdot D_{\boldsymbol{\xi}' \boldsymbol{\xi}'}^2 \psi(\boldsymbol{\xi}')$  and  $\|\boldsymbol{\xi}'\|_{\ell^2} \leq h_T$ , we infer that

$$f(\boldsymbol{\xi}') \geq f(\mathbf{0}) - h_T \|\nabla_{\boldsymbol{\xi}'} f\|_{L^\infty(V(\mathbf{0}))} \geq 2h_T - h_T^2 M \geq h_T.$$

Since  $\|\mathbf{n}(\boldsymbol{\xi}')\|_{\ell^2} \leq 1 + \|\nabla_{\boldsymbol{\xi}'} \psi(\boldsymbol{\xi}')\|_{\ell^2} \leq 1 + hM \leq 2$ , we infer that

$$d(\hat{\mathbf{x}}_T, T_s\Gamma) = \|\mathbf{n}(\boldsymbol{\xi}')\|_{\ell^2}^{-1} f(\boldsymbol{\xi}') \geq \frac{1}{2} h_T, \quad \forall \mathbf{s} \in T^\Gamma.$$

In addition, we have  $\|\hat{\mathbf{x}}_T - \mathbf{s}\|_{\ell^2} \leq \|\boldsymbol{\xi}'\|_{\ell^2} + |\psi(\boldsymbol{\xi}') + 2h_T| \leq 3h_T + |\psi(\boldsymbol{\xi}')| \leq 4h_T$  since  $\psi(\mathbf{0}) = 0$ ,  $\nabla_{\boldsymbol{\xi}'} \psi(\mathbf{0}) = \mathbf{0}$  and  $hM \leq 1$ .  $\square$

## 6.2 Assumption 2: local cell-agglomeration

Assume that we are given an initial shape-regular (polyhedral) mesh  $\mathcal{T}_h$  (with parameter  $\rho$ ) that satisfies Assumption 1 (with parameter  $\gamma$ ), but that does not satisfy Assumption 2. We now describe a simple local cell-agglomeration procedure to produce a new mesh that is still shape-regular and that satisfies Assumption 1 and Assumption 2. The main idea is that we eliminate any mesh cell in  $\mathcal{T}_h$  that is cut unfavorably by the interface by merging this cell with a neighboring one. An illustration is provided in Figure 5. We consider the partition  $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^\Gamma \cup \mathcal{T}_h^2$ , and picking a value  $\delta \in (0, 1)$  (the precise value of  $\delta$  is determined below), we further partition  $\mathcal{T}_h^\Gamma$  into

$$\mathcal{T}_h^\Gamma = \mathcal{T}_h^{\text{OK}} \cup \mathcal{T}_h^{\text{KO},1} \cup \mathcal{T}_h^{\text{KO},2}, \quad (52)$$

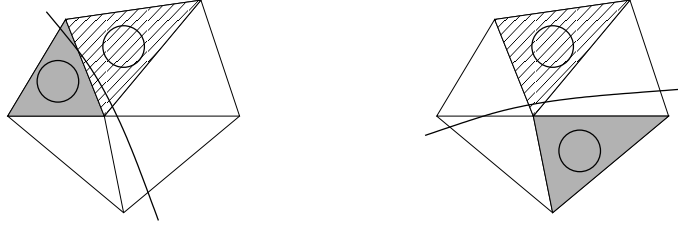


Figure 5: Triangular mesh cell  $T$  (filled by dashes) cut unfavorably by the interface; the companion cell in the agglomeration procedure is shown in gray for two generic situations: on the left, both cells share a face, and on the right, they only share a vertex; in both cases, the companion cell is in  $\Delta(T)$ , and the two balls in the agglomerated cell for Assumption 2 are shown.

where  $T \in \mathcal{T}_h^{\text{OK}}$  iff the condition (6) from Assumption 2 holds true for all  $i \in \{1, 2\}$ , whereas  $T \in \mathcal{T}_h^{\text{KO}, i}$  if the condition fails for  $i \in \{1, 2\}$ . Let us first give two useful lemmas underpinning our local cell-agglomeration procedure.

**Lemma 15** (Partition of  $\mathcal{T}_h^\Gamma$ ). *The subsets  $\mathcal{T}_h^{\text{KO}, 1}$  and  $\mathcal{T}_h^{\text{KO}, 2}$  are disjoint if the mesh-size  $h$  is small enough and if  $\delta \leq \frac{1}{3}\rho$ .*

*Proof.* For  $\mathbf{s} \in \Gamma$  and positive real numbers  $\alpha, \beta$ , we define the strip of length  $\alpha$  and aspect ratio  $\beta$  centered at the point  $\mathbf{s}$  and aligned with the tangent plane  $T_{\mathbf{s}}\Gamma$  as

$$S_\Gamma(\mathbf{s}, \alpha, \beta) := \mathbf{s} + \{\mathbf{t} + \mathbf{n}, \mathbf{t} \in T_{\mathbf{s}}\Gamma, 2\|\mathbf{t}\|_{\ell^2} \leq \alpha, \mathbf{n} \in (T_{\mathbf{s}}\Gamma)^\perp, 2\|\mathbf{n}\|_{\ell^2} \leq \alpha\beta\}.$$

The regularity of  $\Gamma$  implies that, for all  $\lambda \in (0, 1]$ , there is  $\delta(\lambda) > 0$ , so that, for all  $\mathbf{s} \in \Gamma$  and all  $\alpha \in (0, \delta(\lambda)]$ ,

$$\Gamma \cap B(\mathbf{s}, \alpha) \subset S_\Gamma(\mathbf{s}, 2\alpha, \lambda). \quad (53)$$

(Note that the diameter of  $B(\mathbf{s}, \alpha)$  is  $2\alpha$ .) Let  $T \in \mathcal{T}_h^\Gamma$  and let  $\mathbf{s} \in T^\Gamma$ . Recall from mesh regularity (property (i)) that  $B(\mathbf{x}_T, \rho h_T) \subset T$ . Let us apply (53) with  $\lambda = \frac{1}{3}\rho$ . Assume that  $h \leq \delta(\frac{1}{3}\rho)$ . Then  $T^\Gamma \subset \Gamma \cap B(\mathbf{s}, h_T) \subset S_\Gamma(\mathbf{s}, 2h_T, \frac{1}{3}\rho)$ . Elementary geometric considerations show that there is a point  $\tilde{\mathbf{y}}_T \in T$  so that  $B(\tilde{\mathbf{y}}_T, \frac{1}{3}\rho h_T) \subset B(\mathbf{x}_T, \rho h_T) \setminus S_\Gamma(\mathbf{s}, 2h_T, \frac{1}{3}\rho)$ , which implies, in particular, that  $B(\tilde{\mathbf{y}}_T, \frac{1}{3}\rho h_T) \cap \Gamma = \emptyset$ . Therefore,  $B(\tilde{\mathbf{y}}_T, \frac{1}{3}\rho h_T)$  is a subset of either  $T^1$  or  $T^2$ .  $\square$

**Lemma 16** (Finding a suitable neighbor). *Assume that the mesh-size is small enough and take  $\delta = \frac{1}{4}\rho^3$ . Let  $i \in \{1, 2\}$ . For all  $T \in \mathcal{T}_h^{\text{KO}, i}$ , there is a mesh cell in  $\Delta(T)$  such that the condition (6) holds true for  $i$ , i.e., this mesh cell is in  $(\mathcal{T}_h^i \cup \mathcal{T}_h^{\text{OK}} \cup \mathcal{T}_h^{\text{KO}, \bar{i}}) \cap \Delta(T)$ , where  $\bar{i} = 3 - i$  (so that  $\bar{i} = 2$  if  $i = 1$  and  $\bar{i} = 1$  if  $i = 2$ ).*

*Proof.* Fix  $i \in \{1, 2\}$  and let  $T \in \mathcal{T}_h^{\text{KO}, i}$ . Owing to mesh regularity (property (ii)), we have  $T_\rho := \{\mathbf{x} \in \mathbb{R}^d, d(\mathbf{x}, T) \leq \rho h_T\} \subset \Delta(T)$ . Let  $\mathbf{s} \in T^\Gamma$ . Assume that  $h \leq \delta(\frac{1}{4}\rho)$  (see (53)), so that  $\Gamma \cap B(\mathbf{s}, h_T) \subset S_\Gamma(\mathbf{s}, 2h_T, \frac{1}{4}\rho)$ . Since the width of  $S_\Gamma$  is smaller than or equal to  $\frac{1}{2}\rho h_T$ , there is a ball  $B(\mathbf{s}', \frac{1}{4}\rho h_T) \subset T_\rho \cap \Omega^i \setminus S_\Gamma(\mathbf{s}, 2h_T, \frac{1}{4}\rho)$ . Note that  $d(\mathbf{s}', \partial\Omega) \geq d(\mathbf{s}, \partial\Omega) - h \geq h$  since  $d(\Gamma, \partial\Omega) \geq 2h$  and  $\mathbf{s} \in \Gamma$ . Since  $\mathbf{s}' \in T_\rho$ , there is  $T_1 \in \Delta(T)$  s.t.  $\mathbf{s}' \in T_1$ . Using

mesh regularity (property (iv) with  $\alpha = \frac{1}{4}\rho h_T h_{T_1}^{-1} \leq \frac{1}{4}$  owing to property (iii)), we infer that there is  $T_2 \in \mathcal{T}_h$  so that  $T_2 \cap B(\mathbf{s}', \frac{1}{4}\rho h_T)$  has positive  $d$ -measure and there is a ball  $B(\mathbf{x}_{T_2}, \rho \alpha h_{T_2}) \subset T_2 \cap B(\mathbf{s}', \frac{1}{4}\rho h_T)$ . Since  $\rho \alpha = \frac{1}{4}\rho^2 h_T h_{T_1}^{-1} \geq \frac{1}{4}\rho^3 = \delta$  (using again property (iii)), we infer that the mesh cell  $T_2$  satisfies the condition (6) for  $i$ . Moreover,  $T_2 \cap T_\rho$  has positive  $d$ -dimensional measure, so that  $T_2 \in \Delta(T)$ . This concludes the proof.  $\square$

We can now present our local cell-agglomeration procedure. We consider the partition (52) with  $\delta := \frac{1}{4}\rho^3$ , and assume that the mesh-size is small enough so that Lemma 15 and Lemma 16 hold true (note that  $\delta \leq \frac{1}{3}\rho$ ). The procedure is as follows: (1) For all  $T \in \mathcal{T}_h^{\text{KO},1}$ , we choose a neighboring mesh cell  $N_1(T) \in (\mathcal{T}_h^{\text{OK}} \cup \mathcal{T}_h^1 \cup \mathcal{T}_h^{\text{KO},2}) \cap \Delta(T)$  (this is possible owing to Lemma 16). We denote the collection of the cells in  $\mathcal{T}_h^{\text{KO},2}$  chosen in the above step as the subset  $\hat{\mathcal{T}}_h^{\text{KO},2} \subset \mathcal{T}_h^{\text{KO},2}$ . (2) For all  $T \in \mathcal{T}_h^{\text{KO},2} \setminus \hat{\mathcal{T}}_h^{\text{KO},2}$ , we choose a neighboring mesh cell  $N_2(T) \in (\mathcal{T}_h^{\text{OK}} \cup \mathcal{T}_h^2 \cup \mathcal{T}_h^{\text{KO},1}) \cap \Delta(T)$ . (3) For all  $i \in \{1, 2\}$ , let  $\mathcal{N}_i$  be the collection of all the cells in  $\mathcal{T}_h^{\text{OK}} \cup \mathcal{T}_h^i \cup \mathcal{T}_h^{\text{KO},i}$  that have been selected at least once in one of the two previous steps. For all  $T^\sharp \in \mathcal{N}_1 \cup \mathcal{N}_2$ , we define the agglomerated cell

$$T^* := T^\sharp \cup \{T \in \mathcal{T}_h^{\text{KO},1}, N_1(T) = T^\sharp\} \cup \{T \in \mathcal{T}_h^{\text{KO},2} \setminus \hat{\mathcal{T}}_h^{\text{KO},2}, N_2(T) = T^\sharp\}, \quad (54)$$

and we observe that  $T^* \subset \Delta(T^\sharp)$ . We collect all the agglomerated cells in  $\mathcal{T}_h^{\text{agglo}}$ , and we define the new mesh

$$\mathcal{T}_h^* := \left( (\mathcal{T}_h^{\text{OK}} \cup \mathcal{T}_h^1 \cup \mathcal{T}_h^2) \setminus (\mathcal{N}_1 \cup \mathcal{N}_2) \right) \cup \mathcal{T}_h^{\text{agglo}}. \quad (55)$$

*Remark 6.1. (Choice of  $N_i(T)$ )* In Steps 1 and 2, we do not require that  $T$  and  $N_i(T)$  share a face, it is sufficient that they share a point (actually, it is just sufficient that the set  $T \cup N_i(T)$  has a diameter of order  $h_T$ , but we do not explore this further here). Thus, there is some freedom in the choice of  $N_i(T)$ . In practice, one can choose  $N_i(T)$  sharing a face with  $T$  whenever possible.

It is easy to see that the newly created mesh  $\mathcal{T}_h^*$  is still shape-regular and satisfies Assumption 1. Shape regularity follows since the agglomeration of a finite number of shape-regular neighbors remains shape regular, but with a possibly smaller parameter  $\rho^* < \rho$ . Assumption 1 is satisfied since each cell in the original mesh satisfies the assumption and any finite union of cells satisfying this assumption must also satisfy it, but once again with a possibly smaller parameter  $\gamma^* < \gamma$ . Let us finally verify that  $\mathcal{T}_h^*$  also satisfies Assumption 2.

**Lemma 17** (Assumption 2). *Assume that the mesh-size is small enough and that the cell-agglomeration procedure uses the cut parameter  $\delta = \frac{1}{4}\rho^3$ . Then Assumption 2 holds true for the mesh  $\mathcal{T}_h^*$  with  $\delta^* = \frac{1}{3}\rho\delta = \frac{1}{12}\rho^4$ .*

*Proof.* Let  $T^* \in \mathcal{T}_h^*$  be s.t.  $\text{meas}_{d-1}(T \cap \Gamma) > 0$ . Then,  $T^* \in \mathcal{T}_h^{\text{OK}} \setminus (\mathcal{N}_1 \cup \mathcal{N}_2)$  or  $T^* \in \mathcal{T}_h^{\text{agglo}}$ . In the first case,  $T^*$  is also a mesh cell from the original mesh  $\mathcal{T}_h$ , and the definition of  $\mathcal{T}_h^{\text{OK}}$  implies that the condition (6) is satisfied with the cut parameter  $\delta$ , and therefore also with the cut parameter  $\delta^* \leq \delta$ . In the second case where  $T^* \in \mathcal{T}_h^{\text{agglo}}$ , let us assume to fix the ideas that the associated cell  $T^\sharp$  (see (54)) is in  $\mathcal{N}_1$ , so that  $T^\sharp = N_1(T_0^\sharp)$  with



$T_0^\sharp \in \mathcal{T}_h^{\text{KO},1}$ . Owing to Lemma 15, the condition (6) is satisfied in  $T_0^\sharp$  with parameter  $\delta$  and  $i = 2$ , and by construction, this condition is satisfied in  $T^\sharp$  with parameter  $\delta$  and  $i = 1$ . Since  $h_{T^*} \leq h_{\Delta(T^\sharp)} \leq 3 \max_{T' \in \Delta(T^\sharp)} h_{T'} \leq 3\rho^{-1} \min(h_{T^\sharp}, h_{T_0^\sharp})$  owing to mesh regularity (property (iii)), we infer that the condition (6) is satisfied in  $T^*$  with parameter  $\delta^*$  and all  $i \in \{1, 2\}$ .  $\square$

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